Real Analysis Theorems and Definitions

Alexander J. Clarke

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Introduction to Real Analysis, 4th Edition, by Robert G. Bartle and Donald R. Sherbert

Contents

Chapter 1

Preliminaries

1.1 Sets and Functions

Definition 1.1.1. Two sets A and B are said to be **equal,** and we write $A = B$ if they contain the same elements.

Thus, to prove that the sets A and B are equal, we must show that

$$A \subseteq B \text{ and } B \subseteq A$$

Definition 1.1.2.

1. The **union** of sets A and B is the set

$$A \cup B := \{x : x \in A \text{ or } x \in B\}.$$

2. The **intersection** of the sets A and B is the set

$$A \cap B := \{x : x \in A \text{ and } x \in B\}.$$

3. The **complement of B relative to A** is the set

$$A \setminus B := \{x : x \in A \text{ and } x \notin B\}$$

Theorem 1.1.1. If A, B, C are sets, then

1. $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$,

2. $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

Definition 1.1.3. If A and B are nonempty sets, then the **Cartesian product** $A \times B$ of A and B is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$. That is,

$$A \times B := \{(a, b) : a \in A,\ b \in B\}.$$

Definition 1.1.4. Let A and B be sets. Then a **function** from A to B is a set f of ordered pairs in $A \times B$ such that for each $a \in A$ there exists a unique $B \in B$ with $(a, b) \in f$. (In other words, if $(a, b) \in f$ and $(a, b') \in f$, then $b = b'$.)

Definition 1.1.5. If E is a subset of A, then the **direct image** of E under f is the subset $f(E)$ of B given by

$$f(E) := \{f(x) : x \in E\}$$

If H is a subset of B, then the **inverse image** of H under f is the subset $f^{-1}(H)$ of A given by

$$f^{-1}(H) := \{x \in A : f(x) \in H\}$$

Definition 1.1.6. Let $f : A \to B$ be a function from A to B.

1. The function f is said to be **injective** (or to be **one-one**) if whenever $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. If f is an injective function, we also say that f is an **injection**.

2. The function f is said to be **surjective** (or to map A **onto** B) if $f(A) = B$; that is, if the range $R(f) = B$. If f is a surjective function, we also say that f is a **surjection**.

3. If f is both injective and surjective, then f is said to be **bijective**. If f is bijective, we also say that f is a **bijection**.

Definition 1.1.7. If $f : A \to B$ is a bijection of A onto B, then

$$g := \{(b, a) \in B \times A : (a, b) \in f\}$$

is a function on B into A. This function is called the **inverse function** of f, and is denoted by f^{-1}. The function f^{-1} is also called the **inverse** of f.
We can also express the connection between f and its inverse f^{-1} by noting that $D(f) = R(f^{-1})$ and $R(f) = D(f^{-1})$ and that

$$b = f(a) \text{ if and only if } a = f^{-1}(b)$$

Definition 1.1.8. If $f : A \to B$ and $g : B \to C$, and if $R(f) \subseteq D(g) = B$, then the **composite function** $g \circ f$ (note the order!) is the function from A into C defined by

$$(g \circ f)(x) := g(f(x)) \text{ for all } x \in A$$

Theorem 1.1.2. Let $f : A \to B$ and $g : B \to C$ be functions and let H be a subset of C. Then we have

$$(g \circ f)^{-1}(H) = f^{-1}(g^{-1}(H)).$$

Note the reversal in the order of the functions.

1.2 Mathematical Induction

Theorem 1.2.1 (Well-Ordering Property of $\mathbb{N}$). Every nonempty subset of $\mathbb{N}$ has a least element.

A more detailed statement of this property is as follows: If S is a subset of $\mathbb{N}$ and if $S \neq \emptyset$, then there exists $m \in S$ such that $m \leq k$ for all $k \in S$.

Theorem 1.2.2 (Principle of Mathematical Induction). Let S be a subset of $\mathbb{N}$ that possesses the two properties:

1. The number $1 \in S$.

2. For every $k \in \mathbb{N}$, if $k \in S$, then $k+1 \in S$.

Then we have $S = \mathbb{N}$.

Theorem 1.2.3 (Principle of Mathematical Induction (second version)). Let $n_0 \in \mathbb{N}$ and let $P(n)$ be a statement for each natural number $n \geq n_0$. Suppose that:

1. The statement $P(n_0)$ is true.

2. For all $k \geq n_0$, the truth of $P(k)$ implies the truth of $P(k+1)$.

Then $P(n)$ is true for all $n \geq n_0$.

Theorem 1.2.4 (Principle of Strong Induction). Let S be a subset of $\mathbb{N}$ such that

1. $1 \in S$.

2. For every $k \in \mathbb{N}$, if $\{1, 2, \ldots\} \subseteq S$, then $k+1 \in S$.

Then $S = \mathbb{N}$.

1.3 Finite and Infinite Sets

Definition 1.3.1.

1. The empty set $\emptyset$ is said to have 0 **elements**.

2. If $n \in \mathbb{N}$, a set S is said to have n **elements** if there exists a bijection from the set $\mathbb{N}_n := \{1, 2, \ldots, n\}$ onto S.

3. A set S is said to be **finite** if it is either empty or it has n elements for some $n \in \mathbb{N}$.

4. A set S is said to be **infinite** if it is not finite.

Theorem 1.3.1 (Uniqueness Theorem). If S is a finite set, then the number of elements is S is a unique number in $\mathbb{N}$.

Theorem 1.3.2. The set $\mathbb{N}$ of natural numbers is an infinite set.

Theorem 1.3.3.

1. If A is a set with m elements and B is a set with n elements and if $A \cap B = \emptyset$, then $A \cup B$ has $m + n$ elements.

2. If A is a set with $m \in \mathbb{N}$ elements and $C \subseteq A$ is a set with 1 element, then $A \setminus C$ is a set with $m - 1$ elements.

3. If C is an infinite set and B is a finite set, then $C \setminus B$ is an infinite set.

Theorem 1.3.4. Suppose that S and T are sets and that $T \subseteq S$.

1. If S is a finite set, then T is a finite set.

2. If T is an infinite set, then S is an infinite set.

Definition 1.3.2.

1. A set S is said to be **denumerable** (or **countably infinite**) if there exists a bijection of $\mathbb{N}$ onto S.

2. A set S is said to be **countable** if it is either finite or denumerable.

3. A set S is said to be **uncountable** if it is not countable.

Theorem 1.3.5. The set $\mathbb{N} \times \mathbb{N}$ is denumerable.

Theorem 1.3.6. Suppose that S and T are sets and that $T \subseteq S$.

1. If S is a countable set, then T is a countable set.

2. If T is an uncountable set, then S is an uncountable set.

Theorem 1.3.7. The following statements are equivalent:

1. S is a countable set.

2. There exists a surjection of $\mathbb{N}$ onto S.

3. There exists an injection of S into $\mathbb{N}$.

Theorem 1.3.8. The set $\mathbb{Q}$ of all rational numbers is denumerable.

Theorem 1.3.9. If A_m is a countable set for each $m \in \mathbb{N}$, then the union $A := \bigcup\limits_{m=1}^{\infty} A_m$ is countable.

Theorem 1.3.10 (Cantor's Theorem). If A is any set, then there is no surjection of A onto the set $\mathcal{P}(A)$ of all subsets of A.

Chapter 2

The Real Numbers

2.1 The Algebraic and Order Properties of $\mathbb{R}$

Theorem 2.1.1 (**Algebraic Properties of** $\mathbb{R}$**).** On the set $\mathbb{R}$ of real numbers there are two binary operations, denoted by $+$ and $\cdot$ and called **addition** and **multiplication**, respectively. These operations satisfy the following properties:

(A1) $a + b = b + a\ \forall\ a, b \in \mathbb{R}$. (*commutative property of addition*);

(A2) $(a + b) + c = a + (b + c)\ \forall\ a, b, c \in \mathbb{R}$ (*associative property of addition*);

(A3) There exists and element 0 in $\mathbb{R}$ such that $0 + a = a$ and $a + 0 = a$ for all $a \in \mathbb{R}$ (*existence of a zero element*);

(A4) for each $a \in \mathbb{R}$ there exists and element $-a \in \mathbb{R}$ such that $a + (-a) = 0$ and $(-a) + a = 0$ (*existence of negative elements*);

(M1) $a \cdot b = b \cdot a\ \forall\ a, b \in \mathbb{R}$ (*commutative property of multiplication*);

(M2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)\ \forall\ a, b, c \in \mathbb{R}$ (*associative property of multiplication*);

(M3) There exists an element $1 \in \mathbb{R}$ *distinct from* 0 such that $1 \cdot a = a$ and $a \cdot 1 = a\ \forall\ a \in \mathbb{R}$ (*existence of a unit element*);

(M4) for each $a \neq 0 \in \mathbb{R}$, there exists an element $1/a \in \mathbb{R}$ such that $a \cdot (1/a) = 1$ and $(1/a) \cdot a = 1$ (*existence of reciprocals*);

(D) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(b + c) \cdot a = (b \cdot a) + (c \cdot a)\ \forall\ a, b, c \in \mathbb{R}$ (*distributive property of multiplication over addition*).

Theorem 2.1.2.

1. If z and a are elements in $\mathbb{R}$ with $z + a = a$, then $z = 0$.

2. If u and $b \neq 0$ are elements in $\mathbb{R}$ with $u \cdot b = b$, then $u = 1$.

3. If $a \in \mathbb{R}$, then $a \cdot 0 = 0$.

Theorem 2.1.3.

1. If $a \neq 0$ and $b \in \mathbb{R}$ are such that $a \cdot b = 1$, then $b = 1/a$.
2. If $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

Theorem 2.1.4. There does not exists a rational number r such that $r^2 = 2$.

Definition 2.1.1 (The Order Properties of $\mathbb{R}$). There is a nonempty subset $\mathbb{P}$ of $\mathbb{R}$, called the set of **positive real numbers**, that satisfies the following properties:

1. If a, b belong to $\mathbb{P}$, then $a + b$ belongs to $\mathbb{P}$.
2. If a, b belong to $\mathbb{P}$, then ab belongs to $\mathbb{P}$.
3. If a belongs to $\mathbb{R}$, then exactly one of the following holds:
$$a \in \mathbb{P}, \quad a = 0, \quad -a \in \mathbb{P}$$
(This condition is usually called the **Trichotomy Property**.)

Definition 2.1.2. Let a, b be elements of $\mathbb{R}$.

1. If $a - b \in \mathbb{P}$, then we write $a > b$ or $b < a$.
2. If $a - b \in \mathbb{P} \cup \{0\}$, then we write $a \geq b$ or $b \leq a$.

Theorem 2.1.5. Let a, b, c be any elements of $\mathbb{R}$.

1. If $a > b$ and $b > c$, then $a > c$.
2. If $a > b$, then $a + c > b + c$.
3. If $a > b$ and $c > 0$, then $ca > cb$.
If $a > b$ and $c < 0$, then $ca < cb$.

Theorem 2.1.6.

1. If $a \in \mathbb{R}$ and $a \neq 0$, then $a^2 > 0$.
2. $1 > 0$.
3. If $n \in \mathbb{N}$, then $n > 0$

Theorem 2.1.7. If $a \in \mathbb{R}$ is such that $0 \leq a < \varepsilon$ for every $\varepsilon > 0$, then $a = 0$.

Theorem 2.1.8. If $ab > 0$, then either

1. $a > 0$ and $b > 0$, or
2. $a < 0$ and $b < 0$.

Corollary 2.1.1. If $ab < 0$, then either

1. $a < 0$ and $b > 0$, or
2. $a > 0$ and $b < 0$.

Definition 2.1.3 (Bernoulli's Inequality). If $x > -1$, then
$$(1 + x)^n \geq 1 + nx \ \forall \ n \in \mathbb{N}$$

2.2 Absolute Value and the Real Line

Definition 2.2.1. The **absolute value** of a real number a, denoted by $|a|$, is defined by

$$|a| := \begin{cases} a & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ -a & \text{if } a < 0. \end{cases}$$

Theorem 2.2.1.

1. $|ab| = |a||b|$ for all $a, b \in \mathbb{R}$.
2. $|a|^2 = a^2$ for all $a \in \mathbb{R}$.
3. If $c \geq 0$, then $|a| \leq c$ if and only if $-c \leq a \leq c$.
4. $-|a| \leq a \leq |a|$ for all $a \in \mathbb{R}$.

Theorem 2.2.2 (Triangle Inequality). If $a, b \in \mathbb{R}$, then $|a + b| \leq |a| + |b|$.

Corollary 2.2.1. If $a, b \in \mathbb{R}$, then

1. $||a| - |b|| \leq |a - b|$,
2. $|a - b| \leq |a| + |b|$.

Corollary 2.2.2. If $a_1, a_2, \ldots, a_n$ are any real numbers, then

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$$

Definition 2.2.2. Let $a \in \mathbb{R}$ and $\varepsilon > 0$. Then the ε-**neighborhood** of a is the set $V_\varepsilon(a) := \{x \in \mathbb{R} : |x - a| < \varepsilon\}$.

2.3 The Completeness Property of $\mathbb{R}$

Definition 2.3.1. Let S be a nonempty subset of $\mathbb{R}$.

1. The set S is said to be **bounded above** if there exists a number $u \in \mathbb{R}$ such that $s \leq u$ for all $s \in S$. Each such number u is called an **upper bound** of S.
2. The set S is said to be **bounded below** if there exists a number $w \in \mathbb{R}$ such that $w \leq s$ for all $s \in S$. Each such number w is called a **lower bound** of S.
3. A set is said to be **bounded** if it is both bounded above and bounded below. A set is said to be **unbounded** if it is not bounded.

Definition 2.3.2. Let S be a nonempty subset of $\mathbb{R}$.

1. If S is bounded above, then a number u is said to be a **supremum** (or a **least upper bound**) of S if it satisfies the conditions:

(a) u is an upper bound of S, and

(b) if v is any upper bound of S, then $u \leq v$.

2. If S is bounded below, then a number w is said to be an **infimum** (or a **greatest lower bound**) of S if it satisfies the conditions:

(a) w is a lower bound of S, and

(b) if t is any lower bound of S, then $t \leq w$.

Lemma 2.3.1. A number u is the supremum of a nonempty subset S of $\mathbb{R}$ if and only if u satisfies the conditions:

1. $s \leq u$ for all $s \in S$,

2. if $v < u$, then there exists $s' \in S$ such that $v < s'$.

Lemma 2.3.2. An upper bound u of a nonempty set S in $\mathbb{R}$ is the supremum of S if and only if for every $\varepsilon > 0$ there exists an $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$.

Theorem 2.3.1 (The Completeness Property of $\mathbb{R}$). Every nonempty set of real numbers that has an upper bound also has a supremum in $\mathbb{R}$. (This property is also called the **Supremum Property of** $\mathbb{R}$).

2.4 Applications of the Supremum Property

Theorem 2.4.1 (Archimedian Property). If $x \in \mathbb{R}$, then there exists $n_x \in \mathbb{N}$ such that $x \leq n_x$.

Corollary 2.4.1. If $S := \{1/n : n \in \mathbb{N}\}$, then $\inf S = 0$.

Corollary 2.4.2. If $t > 0$, there exists $n_t \in \mathbb{N}$ such that $0 < 1/n_t < t$.

Corollary 2.4.3. If $y > 0$, there exists $n_y \in \mathbb{N}$ such that $n_y - 1 \leq y \leq n_y$.

Theorem 2.4.2. There exists a positive real number x such that $x^2 = 2$.

Theorem 2.4.3 (The Density Theorem). If x and y are any real numbers with $x < y$, then there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$.

Corollary 2.4.4. If x and y are real numbers with $x < y$, then there exists an irrational number z such that $x < z < y$.

2.5 Intervals

Definition 2.5.1. If $a, b \in \mathbb{R}$ satisfy $a < b$, then the **open interval** determined by a and b is the set

$$(a, b) := \{x \in \mathbb{R} : a < x < b\}$$

The points a and b are called the **endpoints** of the interval.

Definition 2.5.2. If both endpoints a and b are adjoined to an open interval, then we obtain the **closed interval** determined by a and b; namely, the set

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$$

Definition 2.5.3. The two **half-open** (or **half-closed**) intervals determined by a and b are $[a, b)$, which includes the endpoint a, and $(a, b]$, which includes the endpoint b.

Definition 2.5.4. The **length** of an interval (a, b) is defined by $b - a$.

Theorem 2.5.1 (Characterization Theorem). If S is a subset of $\mathbb{R}$ that contains at least two points and has the property

$$\text{if} \quad x, y \in S \quad \text{and} \quad x < y, \quad \text{then} \quad [x, y] \subseteq S,$$

then S is an interval.

Theorem 2.5.2 (Nested Intervals Property). If $I_n = [a_n, b_n],\ n \in \mathbb{N}$, is a nested sequence of closed bounded intervals, then there exists a number $\xi \in \mathbb{R}$ such that $\xi \in I_n$ for all $n \in \mathbb{N}$.

Theorem 2.5.3. If $I_n := [a_n, b_n],\ n \in \mathbb{N}$, is a nested sequence of closed, bounded intervals such that the lengths $b_n - a_n$ of I_n satisfy

$$\inf\{b_n - a_n : n \in \mathbb{N}\} = 0,$$

then the number ξ contained in I_n for all $n \in \mathbb{N}$ is unique.

Theorem 2.5.4. The set $\mathbb{R}$ of real numbers is not countable.

Theorem 2.5.5. The unit interval $[0, 1] := \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ is not countable.

Chapter 3

Sequences and Series

3.1 Sequences and Their Limits

Definition 3.1.1. A **sequence of real numbers** (or a **sequence in** $\mathbb{R}$) is a function defined on the set $\mathbb{N} = \{1, 2, \dots\}$ of natural numbers whose range is contained in the set $\mathbb{R}$ of real numbers.

Definition 3.1.2. A sequence $X = (x_n)$ in $\mathbb{R}$ is said to **converge** to $x \in \mathbb{R}$, or x is said to be a **limit** of (x_n), if for every $\varepsilon > 0$ there exists a natural number $K(\varepsilon)$ such that for all $n \geq K(\varepsilon)$, the terms x_n satisfy $|x_n - x| < \varepsilon$.
If a sequence has a limit, we say that the sequence is **convergent**; if it has no limit, we say that the sequence is **divergent**.

Theorem 3.1.1 (Uniqueness of Limits). A sequence in $\mathbb{R}$ can have at most one limit.

Theorem 3.1.2. Let $X = (x_n)$ be a sequence of real numbers, and let $x \in \mathbb{R}$. The following statements are equivalent:

1. X converges x.
2. For every $\varepsilon > 0$, there exists a natural number K such that for all $n \geq K$, the terms x_n satisfy $|x_n - x| < \varepsilon$.
3. For every $\varepsilon > 0$, there exists a natural number K such that for all $n \geq K$, the terms x_n satisfy $x - \varepsilon < x_n < x + \varepsilon$.
4. For every ε-neighborhood $V_\varepsilon(x)$ of x, there exists a natural number K such that for all $n \geq K$, the terms x_n belong to $V_\varepsilon(x)$.

Definition 3.1.3. If $X = (x_1, x_2, \dots, x_n, \dots)$ is a sequence of real numbers and if m is a given natural number, then the m-**tail** of X is the sequence

$$X_m := (x_{m+n} : n \in \mathbb{N}) = (x_{m+1}, x_{m+2}, \dots)$$

Theorem 3.1.3. Let $X = (x_n : n \in \mathbb{N})$ be a sequence of real numbers and let $m \in \mathbb{N}$. Then the m-tail $X_m = (x_{m+n} : n \in \mathbb{N})$ of X converges if and only if X converges. In this case, $\lim X_m = \lim X$.

Theorem 3.1.4. Let (x_n) be a sequence of real numbers and let $x \in \mathbb{R}$. If (a_n) is a sequence of positive real numbers with $\lim(a_n) = 0$ and if for some constant $C > 0$ and some $m \in \mathbb{N}$ we have

$$|x_n - x| \leq C a_n \qquad \text{for all} \qquad n \geq m$$

then it follows that $\lim(x_n) = x$.

3.2 Limit Theorems

Definition 3.2.1. A sequence $X = (x_n)$ of real numbers is said to be **bounded** if there exists a real number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Thus, the sequence (x_n) is bounded if and only if the set $\{x_n : n \in \mathbb{N}\}$ of its values is a bounded subset of $\mathbb{R}$.

Theorem 3.2.1. A convergent sequence of real numbers is bounded.

Theorem 3.2.2. Let $X = (x_n)$ and $Y = (y_n)$ be sequences of real numbers that converge to x and y, respectively, and let $c \in \mathbb{R}$.

1. Then the sequences $X + Y, X - Y, X \cdot Y$, and cX converge to $x + y, x - y, xy$, and cx, respectively.

2. If $X = (x_n)$ converges to x and $Z = (z_n)$ is a sequence of nonzero real numbers that converges to z and if $z \neq 0$, then the quotient sequence X/Z converges to x/z.

Theorem 3.2.3. If $X = (x_n)$ is a convergent sequence of real numbers and if $x_n \geq 0$ for all $n \in \mathbb{N}$, then $x = \lim(x_n) \geq 0$.

Theorem 3.2.4. If $X = (x_n)$ and $Y = (y_n)$ are convergent sequences of real numbers and if $x_n \leq y_n$ for all $n \in \mathbb{N}$, then $\lim(x_n) \leq \lim(y_n)$.

Theorem 3.2.5. If $X = (x_n)$ is a convergent sequence and if $a \leq x_n \leq b$ for all $n \in \mathbb{N}$, then $a \leq \lim(x_n) \leq b$.

Theorem 3.2.6 (Squeeze Theorem). Suppose that $X = (x_n), Y = (y_n)$, and $Z = (z_n)$ are sequences of real numbers such that

$$x_n \leq y_n \leq z_n \quad \text{for all} \quad n \in \mathbb{N}$$

and that $\lim(x_n) = \lim(z_n)$. Then $Y = (y_n)$ is convergent and

$$\lim(x_n) = \lim(y_n) = \lim(z_n).$$

Theorem 3.2.7. Let the sequence $X = (x_n)$ converge to x. Then the sequence $(|x_n|)$ of absolute values converges to $|x|$. That is, if $x = \lim(x_n)$, then $|x| = \lim(|x_n|)$.

Theorem 3.2.8. Let $X = (x_n)$ be a sequence of real numbers that converges to x and suppose that $x_n \geq 0$. Then the sequence $(\sqrt{x_n})$ of positive square roots converges and $\lim(\sqrt{x_n}) = \sqrt{x}$.

Theorem 3.2.9. Let (x_n) be a sequence of positive real numbers such that $L := \lim(x_{n+1}/x_n)$ exists. If $L < 1$, then (x_n) converges and $\lim(x_n) = 0$.

3.3 Monotone Sequences

Definition 3.3.1. Let $X = (x_n)$ be a sequence of real numbers. We say that X is **increasing** if it satisfies the inequalities

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \ldots$$

We say that X is **decreasing** if it satisfies the inequalities

$$x_1 \geq x_2 \geq \cdots \geq x_n \geq x_{n+1} \geq \ldots$$

We say that X is **monotone** if it is either increasing or decreasing.

Theorem 3.3.1 (**Monotone Convergence Theorem**)**.** A monotone sequence of real numbers is convergent if and only if it is bounded. Further:

1. If $X = (x_n)$ is a bounded increasing sequence, then

$$\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\}$$

2. If $Y = (y_n)$ is a bounded decreasing sequence, then

$$\lim(y_n) = \inf\{y_n : n \in \mathbb{N}\}$$

3.4 Subsequences and the Bolzano-Wierstrass Theorem

Definition 3.4.1. Let $X = (x_n)$ be a sequence of real numbers and let $n_1 < n_2 < \cdots < n_k < \ldots$ be a strictly increasing sequence of natural numbers. Then the sequence $X' = (x_{n_k})$ given by

$$(x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots)$$

is called a **subsequence** of X.

Theorem 3.4.1. If a sequence $X = (x_n)$ of real numbers converges to a real number x, then any subsequence $X' = (x_{n_k})$ of X also converges to x.

Theorem 3.4.2. Let $X = (x_n)$ be a sequence of real numbers. Then the following are equivalent:

1. The sequence $X = (x_n)$ does not converge to $x \in \mathbb{R}$.

2. There exists an $\varepsilon_0 > 0$ such that for any $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $n_k \geq k$ and $|x_{n_k} - x| \geq \varepsilon_0$.

3. There exists an $\varepsilon_0 > 0$ and a subsequence $X' = (x_{n_k})$ of X such that $|x_{n_k} - x| \geq \varepsilon_0$ for all $k \in \mathbb{N}$.

Theorem 3.4.3 (**Divergence Criteria**)**.** If a sequence $X = (x_n)$ of real numbers has either of the following properties, then X is divergent.

1. X has two convergent subsequences $X' = (x_{n_k})$ and $X'' = (x_{r_k})$ whose limits are not equal.

2. X is unbounded.

Theorem 3.4.4 (Monotone Subsequence Theorem). If $X = (x_n)$ is a sequence of real numbers, then there is a subsequence of X that is monotone.

Theorem 3.4.5 (The Bolzano-Wierstrass Theorem). A bounded sequence of real numbers has a convergent subsequence.

Theorem 3.4.6. Let $X = (x_n)$ be a bounded sequence of real numbers and let $x \in \mathbb{R}$ have the property that every convergent subsequence of X converges to x. Then the sequence X converges to x.

Definition 3.4.2. Let $X = (x_n)$ be a bounded sequence of real numbers.

1. The **limit superior** of (x_n) is the infimum of the set V of $v \in \mathbb{R}$ such that $v < x_n$ for at most a finite number of $n \in \mathbb{N}$. It is denoted by

$$\limsup(x_n) \quad \text{or} \quad \limsup X \quad \text{or} \quad \overline{\lim}(x_n)$$

2. The **limit inferior** of (x_n) is the supremum of the set of $w \in \mathbb{R}$ such that $x_m < w$ for at most a finite number of $m \in \mathbb{N}$. It is denoted by

$$\liminf(x_n) \quad \text{or} \quad \liminf X \quad \text{or} \quad \overline{\lim}(x_n)$$

Theorem 3.4.7. If (x_n) is a bounded sequence of real numbers, then the following statements for a real number x^* are equivalent.

1. $x^* = \limsup(x_n)$.

2. If $\varepsilon > 0$, there are at most a finite number of $n \in \mathbb{N}$ such that $x^* + \varepsilon < x_n$, but an infinite number of $n \in \mathbb{N}$ such that $x^* - \varepsilon < x_n$.

3. If $u_m = \sup\{x_n : n \geq m\}$, then $x^* = \inf\{u_m : m \in \mathbb{N}\} = \lim(u_m)$.

4. If S is the set of subsequential limits of (x_n), then $x^* = \sup S$.

Theorem 3.4.8. A bounded sequence (x_n) is convergent if and only if $\limsup(x_n) = \liminf(x_n)$.

3.5 The Cauchy Criterion

Theorem 3.5.1. A sequence $X = (x_n)$ of real numbers is said to be a **Cauchy sequence** if for every $\varepsilon > 0$ there exists a natural number $H(\varepsilon)$ such that for all natural numbers $n, m \geq H(\varepsilon)$, the terms x_n, x_m satisfy $|x_n - x_m| < \varepsilon$.

Lemma 3.5.1. If $X = (x_n)$ is a convergent sequence of real numbers, then X is a Cauchy sequence.

Lemma 3.5.2. A Cauchy sequence of real numbers is bounded.

Theorem 3.5.2 (**Cauchy Convergence Criterion**). A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Definition 3.5.1. We say that a sequence $X = (x_n)$ of real numbers is **contractive** if there exists a constant C, $0 < C, 1$, such that

$$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|$$

for all $n \in \mathbb{N}$. The number C is called the **constant** of the contractive sequence.

Theorem 3.5.3. Every contractive sequence is a Cauchy sequence, and therefore is convergent.

Corollary 3.5.1. If $X := (x_n)$ is a contractive sequence with constant $C, 0 < C < 1$, and if $x^* := \lim X$, then

1. $|x^* - x_n| \leq \frac{C^{n-1}}{1-C}|x_2 - x_1|$,
2. $|x^* - x_n| \leq \frac{C}{1-C}|x_n - x_{n-1}|$.

3.6 Properly Divergent Sequences

Definition 3.6.1. Let (x_n) be a sequence of real numbers.

1. We say that (x_n) **tends to** $\pm\infty$, and write $\lim(x_n) = +\infty$, if for every $\alpha \in \mathbb{R}$ there exists a natural number $K(\alpha)$ such that if $n \geq K(\alpha)$, then $x_n > \alpha$.
2. We say that (x_n) **tends to** $-\infty$, and write $\lim(x_n) = -\infty$, if for every $\beta \in \mathbb{R}$ there exists a natural number $K(\beta)$ such that if $n \geq K(\beta)$, then $x_n < \beta$.

We say that (x_n) is **properly divergent** in case we have either $\lim(x_n) = +\infty$, or $\lim(x_n) = -\infty$.

Theorem 3.6.1. A monotone sequence of real numbers is properly divergent if and only if it is unbounded.

1. If (x_n) is an unbounded increasing sequence, then $\lim(x_n) = +\infty$.
2. If (x_n) is an unbounded decreasing sequence, then $\lim(x_n) = -\infty$.

Theorem 3.6.2. Let (x_n) and (y_n) be two sequences of real numbers and suppose that

$$x_n \leq y_n \quad \text{for all} \quad n \in \mathbb{N}$$

1. If $\lim(x_n) = +\infty$, then $\lim(y_n) = +\infty$.
2. If $\lim(y_n) = -\infty$, then $\lim(x_n) = -\infty$.

Theorem 3.6.3. Let (x_n) and (y_n) be two sequences of positive real numbers and suppose that for some $L \in \mathbb{R}, L > 0$, we have

$$\lim(x_n/y_n) = L$$

Then $\lim(x_n) = +\infty$ if an only if $\lim(y_n) = +\infty$.

3.7 Introduction to Infinite Series

Definition 3.7.1. If $X := (x_n)$ is a sequence in $\mathbb{R}$, then the **infinite series** (or simply the **series**) **generated by** X is the sequence $S := (s_k)$ defined by

$$\begin{aligned}
s_1 &:= x_1 \\
s_2 &:= s_1 + s_2 && (= x_1 + x_2) \\
&\cdots \\
s_k &:= s_{k-1} + x_k && (= x_1 + x_2 + \cdots + x_k) \\
&\cdots
\end{aligned}$$

Theorem 3.7.1 (The nth Term Test). If the series $\sum x_n$ converges, then $\lim(x_n) = 0$.

Theorem 3.7.2 (Cauchy Criterion for Series). The series $\sum x_n$ converges if and only if for every $\varepsilon > 0$ there exists $M(\varepsilon) \in \mathbb{N}$ such that if $m > n \geq M(\varepsilon)$, then

$$|s_m - s_n| = |x_{n+1} + x_{n+2} + \cdots + x_m| < \varepsilon$$

Theorem 3.7.3. Let (x_n) be a sequence of nonnegative real numbers. Then the series $\sum x_n$ converges if and only if the sequence $S = (s_k)$ of partial sums is bounded. In this case,

$$\sum_{n=1}^{\infty} x_n = \lim(x_k) = \sup\{s_k : k \in \mathbb{N}\}$$

Theorem 3.7.4 (Comparison Test). Let $X := (x_n)$ and $Y := (y_n)$ be real sequences and suppose that for some $K \in \mathbb{N}$ we have

$$0 \leq x_n \leq y_n \quad \text{for} \quad n \geq K$$

1. Then the convergence of $\sum y_n$ implies the convergence of $\sum x_n$.

2. The divergence of $\sum x_n$ implies the divergence of $\sum y_n$.

Theorem 3.7.5 (Limit Comparison Test). Suppose that $X := (x_n)$ and $Y := (y_n)$ are strictly positive sequences and suppose that the following limit exists in $\mathbb{R}$:

$$r := \lim\left(\frac{x_n}{y_n}\right)$$

1. If $r \neq 0$, then $\sum x_n$ is convergent if and only if $\sum y_n$ is convergent.

2. If $r = 0$ and if $\sum y_n$ is convergent, then $\sum x_n$ is convergent.

3. If $r = \infty$ and $\sum y_n$ diverges, then $\sum x_n$ diverges.

Definition 3.7.2. Let $(a_n) : n \mapsto a(n)$ be a decreasing sequence of strictly positive terms in $\mathbb{R}$ which converges with a limit of zero. That is, for every n in the domain of $(a_n) : a_n > 0,\ a_{n+1} \leq a_n$, and $a_n \to 0$ as $n \to +\infty$. The series $\displaystyle\sum_{n=1}^{\infty} 2^n a(2^n)$ is called the **condensed** form of the series $\displaystyle\sum_{n=1}^{\infty} a_n$.

Theorem 3.7.6 (**Cauchy Condensation Test**). Let $(a_n) : n \mapsto a(n)$ be a decreasing sequence of strictly positive terms in $\mathbb{R}$ which converges with a limit of zero. That is, for every n in the domain of $(a_n) : a_n > 0,\ a_n \geq a_{n+1}$, and $a_n \to 0$ as $n \to +\infty$. Then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges if and only if the condensed series $\displaystyle\sum_{n=1}^{\infty} 2^n a(2^n)$ converges.

Theorem 3.7.7 (**Cauchy Ratio Test**). Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be a series and $a_n > 0$ for all $n \in \mathbb{N}$, and suppose the following limit exists in $\mathbb{R}$:

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

1. If $L < 1$, then the series converges absolutely.

2. If $L > 1$, then the series is divergent.

3. If $L = 1$ or the limit does not exist, then the test is inconclusive.

Theorem 3.7.8 (**Kummer's Test**). Let $\sum a_n$ be a positive term series.

1. If there exists a positive term sequence b_n, $\alpha > 0$, and $\mathbb{N} \in \mathbb{N}$ such that $\dfrac{a_n}{a_{n+1}} \cdot b_n - b_{n+1} \geq \alpha,\ \forall\ n \geq N$, then $\sum a_n$ converges.

2. If $\dfrac{a_n}{a_{n+1}} \cdot b_n - b_{n+1} \leq 0,\ \forall\ n \geq N$, and if $\sum \frac{1}{b_n}$ diverges, then $\sum a_n$ diverges.

Theorem 3.7.9 (**Gauss' Test**). If $\sum a_n$ is a positive term series, and if there exists a bounded sequence b_n such that $\forall\ n \geq N$, $\dfrac{a_n}{a_{n+1}} = 1 + \dfrac{L}{n} + \dfrac{b_n}{n^2}$, then

1. If $L > 1$, then $\sum a_n$ converges.

2. If $L \leq 1$, then $\sum a_n$ diverges.

Chapter 4

Limits

4.1 Limits of Functions

Definition 4.1.1. Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is a **cluster point** of A if for every $\delta > 0$ there exists at least one point $x \in A,\ x \neq c$ such that $|x - c| < \delta$.

This definition is rephrased in the language of neighborhoods as follows: A point c is a cluster point of the set A if every δ-neighborhood $V_\delta(c) = (c - \delta, c + \delta)$ of c contains at least one point of A distinct from c.

Theorem 4.1.1. A number $c \in \mathbb{R}$ is a cluster point of a subset A of $\mathbb{R}$ if and only if there exists a sequence (a_n) in A such that $\lim(a_n) = c$ and $a_n \neq c$ for all $n \in \mathbb{N}$.

Definition 4.1.2. Let $A \subseteq \mathbb{R}$, and let c be a cluster point of A. For a function $f : A \to \mathbb{R}$, a real number L is said to be a **limit of f at c** if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

Theorem 4.1.2. If $f : A \to \mathbb{R}$ and if c is a cluster point of A, then f can have only one limit at c.

Theorem 4.1.3. Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. Then the following statements are equivalent.

1. $\lim\limits_{x \to c} = L$.

2. Given any ε-neighborhood $V_\varepsilon(L)$ of L, there exists a δ-neighborhood $V_\delta(c)$ of c such that if $x \neq c$ is any point in $V_\delta(c) \cap A$, then $f(x)$ belongs to $V_\varepsilon(L)$.

Theorem 4.1.4 (**Sequential Criterion**)**.** Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. Then the following are equivalent.

1. $\lim\limits_{x \to c} f = L$.

2. For every sequence (x_n) in A that converges to c such that $x_n \neq c$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to L.

Theorem 4.1.5 (**Divergence Criteria**)**.** Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A.

1. If $L \in \mathbb{R}$, then f does **not** have limit L at c if and only if there exists a sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ such that the sequence (x_n) converges to c but the sequence $(f(x_n))$ does **not** converge to L.

2. The function f does **not** have a limit at c if and only if there exists a sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ such that the sequence (x_n) converges to c but the sequence $(f(x_n))$ does **not** converge in $\mathbb{R}$.

Let the **signum function** sgn be defined by

$$\operatorname{sgn}(x) := \begin{cases} +1 & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -1 & \text{for } x < 0. \end{cases}$$

4.2 Limit Theorems

Definition 4.2.1. Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. We say that f is **bounded on a neighborhood of** c if there exists a δ-neighborhood $V_\delta(c)$ of c and a constant $M > 0$ such that we have $|f(x)| \leq M$ for all $x \in A \cap V_\delta(c)$.

Theorem 4.2.1. If $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ has a limit at $c \in \mathbb{R}$, then f is bounded on some neighborhood of c.

Theorem 4.2.2. Let $A \subseteq \mathbb{R}$ and let f and g be functions defined on A to $\mathbb{R}$. We define the **sum** $f + g$, the **difference** $f - g$, and the **product** fg on A to $\mathbb{R}$ to be the functions given by

$$(f + g)(x) := f(x) + g(x),$$

$$(f - g)(x) := f(x) - g(x),$$

$$(fg)(x) := f(x)g(x)$$

for all $x \in A$. Further, if $b \in \mathbb{R}$, we define the **multiple** bf to be the function given by

$$(bf)(x) := bf(x) \quad \text{for all} \quad x \in A$$

Finally, if $h(x) \neq 0$ for $x \in A$, we define the **quotient** f/h to be the function given by

$$\left(\frac{f}{h}\right)(x) := \frac{f(x)}{h(x)} \quad \text{for all} \quad x \in A$$

Theorem 4.2.3. let $A \subseteq \mathbb{R}$, let f and g be functions on A to $\mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. Further, let $b \in \mathbb{R}$.

1. If $\lim_{x\to c} f = L$ and $\lim_{x\to c} g = M$, then

$$\lim_{x\to c}(f+g) = L+M,$$
$$\lim_{x\to c}(f-g) = L-M,$$
$$\lim_{x\to c}(fg) = LM,$$
$$\lim_{x\to c}(bf) = bL.$$

2. If $h : A \to \mathbb{R}$, if $h(x) \neq 0$ for all $x \in A$, and if $\lim_{x\to c} h = H \neq 0$, then

$$\lim_{x\to c}\left(\frac{f}{h}\right) = \frac{L}{H}$$

Theorem 4.2.4. Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. If

$$a \leq f(x) \leq b \quad \text{for all} \quad x \in A,\ x \neq c,$$

and if $\lim_{x\to c} f$ exists, then $a \leq \lim_{x\to c} f \leq b$.

Theorem 4.2.5 (Squeeze Theorem). Let $A \subseteq \mathbb{R}$, let $f, g, h : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. If

$$f(x) \leq g(x) \leq h(x) \quad \text{for all} \quad x \in A,\ x \neq c,$$

and if $\lim_{x\to c} f = L = \lim_{x\to c} h$, then $\lim_{x\to c} g = L$.

Theorem 4.2.6. Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A. If

$$\lim_{x\to c} f > 0 \quad \left[\textit{respectively, } \lim_{x\to c} f < 0\right],$$

then there exists a neighborhood $V_\delta(c)$ of c such that $f(x) > 0$ [respectively, $f(x) < 0$] for all $x \in A \cap V_\delta(c),\ x \neq c$.

4.3 Some Extensions of the Limit Concept

Definition 4.3.1. Let $A \in \mathbb{R}$ and let $f : A \to \mathbb{R}$.

1. If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (c, \infty) = \{x \in A : x > c\}$, then we say that $L \in \mathbb{R}$ is a **right-hand limit of f at** c and we write

$$\lim_{x\to c^+} f = L \qquad \text{or} \qquad \lim_{x\to c^+} f(x) = L$$

if given any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for all $x \in A$ with $0 < x - c < \delta$, then $|f(x) - L| < \varepsilon$.

2. If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (-\infty, c) = \{x \in A : x < c\}$, then we say that $L \in \mathbb{R}$ is a **left-hand limit of f at c** and we write

$$\lim_{x \to c^-} f = L \qquad \text{or} \qquad \lim_{x \to c^-} f(x) = L$$

if given any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in A$ with $0 < c - x < \delta$, then $|f(x) - L| < \varepsilon$.

Theorem 4.3.1. Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of $A \cap (c, \infty)$. Then the following statements are equivalent:

1. $\lim\limits_{x \to c^+} f = L$.
2. For every sequence (x_n) that converges to c such that $x_n \in A$ and $x_n > c$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to L.

Theorem 4.3.2. Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of both of the sets $A \cap (c, \infty)$ and $A \cap (-\infty, c)$. Then $\lim\limits_{x \to c} f = L$ if and only if $\lim\limits_{x \to c^+} f = L = \lim\limits_{x \to c^-} f$.

Theorem 4.3.3. Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A.

1. We say that f **tends to ∞ as** $x \to c$, and write

$$\lim_{x \to c} f = \infty$$

if for every $\alpha \in \mathbb{R}$ there exists $\delta = \delta(\alpha) > 0$ such that for all $x \in A$ with $0 < |x - c| < \delta$, then $f(x) > \alpha$.

2. We say that f **tends to $-\infty$ as** $x \to c$, and write

$$\lim_{x \to c} f = -\infty$$

if for every $\beta \in \mathbb{R}$ there exists $\delta = \delta(\beta) > 0$ such that for all $x \in A$ with $0 < |x - c| < \delta$, then $f(x) < \beta$.

Theorem 4.3.4. Let $A \subseteq \mathbb{R}$, let $f, g : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. Suppose that $f(x) \le g(x)$ for all $x \in A,\ x \ne c$.

1. If $\lim\limits_{x \to c} f = \infty$, then $\lim\limits_{x \to c} g = \infty$.
2. If $\lim\limits_{x \to c} g = -\infty$, then $\lim\limits_{x \to c} f = -\infty$.

Definition 4.3.2. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (c, \infty) = \{x \in A : x > c\}$, then we say that f **tends to ∞** [respectively, $-\infty$] as $x \to c^+$, and we write

$$\lim_{x \to c^+} f = \infty \quad \left[\text{respectively, } \lim_{x \to c^+} f = -\infty\right]$$

if for every $\alpha \in \mathbb{R}$ there is $\delta = \delta(\alpha) > 0$ such that for all $x \in A$ with $0 < x - c < \delta$, then $f(x) > \alpha$ [respectively, $f(x) < \alpha$]

Definition 4.3.3. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. We say that $L \in \mathbb{R}$ is a **limit of** f **as** $x \to \infty$, and write

$$\lim_{x\to\infty} f = L \quad \text{or} \quad \lim_{x\to\infty} f(x) = L,$$

if given any $\varepsilon > 0$ there exists $K = K(\varepsilon) > \alpha$ such that for any $x > K$, then $|f(x) - L| < \varepsilon$.

Theorem 4.3.5. Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. Then the following statements are equivalent:

1. $L = \lim_{x\to\infty} f$.

2. For every sequence (x_n) in $A \cap (a, \infty)$ such that $\lim(x_n) = \infty$, the sequence $(f(x_n))$ converges to L.

Definition 4.3.4. let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Suppose that $(a, \infty) \subseteq A$ for some $a \in A$. We say that f **tends to** ∞ [respectively, $-\infty$] **as** $x \to \infty$, and write

$$\lim_{x\to\infty} f = \infty \qquad \left[\text{respectively, } \lim_{x\to\infty} f = -\infty\right]$$

if given any $\alpha \in \mathbb{R}$ there exists $K = K(\alpha) > \alpha$ such that for any $x > K$, then $f(x) > \alpha$ [respectively, $f(x) < \alpha$].

Theorem 4.3.6. Let $A \in \mathbb{R}$, let $f : A \to \mathbb{R}$, and suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. Then the following statements are equivalent:

1. $\lim_{x\to\infty} = \infty$ [respectively, $\lim_{x\to\infty} f = -\infty$]

2. For every sequence (x_n) in (a, ∞) such that $\lim(x_n) = \infty$, then $\lim(f(x_n)) = \infty$ [respectively, $\lim(f(x_n)) = -\infty$].

Theorem 4.3.7. Let $A \subseteq \mathbb{R}$, let $f, g : A \to \mathbb{R}$, and suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. Suppose further that $g(x) > 0$ for all $x > a$ and that for some $L \in \mathbb{R},\ L \neq 0$, we have

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L.$$

1. If $L > 0$, then $\lim_{x\to\infty} f = \infty$ if and only if $\lim_{x\to\infty} g = \infty$.

2. If $L < 0$, then $\lim_{x\to\infty} f = -\infty$ if and only if $\lim_{x\to\infty} g = \infty$.

Chapter 5

Continuous Functions

5.1 Continuous Functions

Definition 5.1.1. Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in A$. We say that f is **continuous at** c if, given any number $\varepsilon > 0$, there exists $\delta > 0$ such that if x is any point of A satisfying $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.
If f fails to be continuous at c, then we say that f is **discontinuous at** c.

Theorem 5.1.1. A function $f : A \to \mathbb{R}$ is continuous at a point $c \in A$ if and only if given any ε-neighborhood $V_\varepsilon(f(c))$ of $f(c)$ there exists a δ-neighborhood $V_\varepsilon(c)$ of c such that if x is any point of $A \cap V_\delta(c)$, then $f(x)$ belongs to $V_\varepsilon(f(c))$, that is

$$f(A \cap V_\varepsilon(c)) \subseteq V_\varepsilon(f(c))$$

Theorem 5.1.2 (**Sequential Criterion for Continuity**)**.** A function $f : A \to \mathbb{R}$ is continuous at the point $c \in A$ if and only if for every sequence (x_n) in A that converges to c, the sequence $(f(x_n))$ converges to $f(c)$.

Theorem 5.1.3 (**Discontinuity Criterion**)**.** Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in A$. Then f is discontinuous at c if and only if there exists a sequence (x_n) in A such that (x_n) converges to c, but the sequence $(f(x_n))$ does not converge to $f(c)$.

Definition 5.1.2. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. If B is a subset of A, we say that f is **continuous on the set** B if f is continuous at every point of B.

5.2 Combinations of Continuous Functions

Theorem 5.2.1. Let $A \subseteq \mathbb{R}$, let f and g be functions on A to $\mathbb{R}$, and let $b \in \mathbb{R}$. Suppose that $c \in A$ and that f and g are continuous at c.

1. Then $f + g$, $f - g$, fg, and bf are continuous at c.

2. If $h : A \to \mathbb{R}$ is continuous at $c \in A$ and if $h(x) \neq 0$ for all $x \in A$, then the quotient f/h is continuous at c.

Theorem 5.2.2. Let $A \subseteq \mathbb{R}$, let f and g be continuous on A to $\mathbb{R}$, and let $b \in \mathbb{R}$.

1. The functions $f + g$, $f - g$, fg, and bf are continuous on A.
2. If $h : A \to \mathbb{R}$ is continuous on A and $h(x) \neq 0$ for $x \in A$, then the quotient f/h is continuous on A.

Theorem 5.2.3. Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $|f|$ be defined by $|f|(x) := |f(x)|$ for $x \in A$

1. If f is continuous at at point $c \in A$, then $|f|$ is continuous at c.
2. If f is continuous on A, then $|f|$ is continuous on A.

Theorem 5.2.4. Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $f(x) \geq 0$ for all $x \in A$. We let $\sqrt{f}$ be defined for $x \in A$ by $(\sqrt{f})(x) := \sqrt{f(x)}$.

1. If f is continuous at at point $c \in A$, then $\sqrt{f}$ is continuous at c.
2. If f is continuous on A, then $\sqrt{f}$ is continuous on A.

Theorem 5.2.5. Let $A, B \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be functions such that $f(A) \subseteq B$. If f is continuous at a point $c \in A$ and g is continuous at $b = f(c) \in B$, then the composition $g \circ f : A \to \mathbb{R}$ is continuous c.

Theorem 5.2.6. Let $A, B \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, be continuous on A, and let $g : B \to \mathbb{R}$ be continuous on B. If $f(A) \subseteq B$, then the composite function $g \circ f : A \to \mathbb{R}$ is continuous on A.

5.3 Continuous Functions on Intervals

Definition 5.3.1. A function $f : A \to \mathbb{R}$ is said to be **bounded on** A if there exists a constant $M > 0$ such that $|f(x)| \leq M$ for all $x \in A$.

Theorem 5.3.1 (Boundedness Theorem). Let $I := [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f is bounded on I.

Definition 5.3.2. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. We say that f **has an absolute maximum** on A if there is a point $x^* \in A$ such that

$$f(x^*) \geq f(x) \quad \text{for all} \quad x \in A.$$

We say that f **has an absolute minimum** on A if there is a point $x_* \in A$ such that

$$f(x_*) \leq f(x) \quad \text{for all} \quad x \in A.$$

We say that x^* is an **absolute maximum point** for f on A, and that x_* is an **absolute minimum point** for f on A, if they exist.

Theorem 5.3.2 (Maximum-Minimum Theorem). Let $I := [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f has an absolute maximum and an absolute minimum on I.

Theorem 5.3.3 (**Location of Roots Theorem**). Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be continuous on I. If $f(a) < 0 < f(b)$, or if $f(a) > 0 > f(b)$, then there exists a number $c \in (a, b)$ such that $f(c) = 0$.

Theorem 5.3.4 (**Bolzano's Intermediate Value Theorem**). Let I be an interval and let $f : I \to \mathbb{R}$ be continuous on I. If $a, b \in I$ and if $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$, then there exists a point $c \in I$ between a and b such that $f(c) = k$.

Corollary 5.3.1. Let $I = [a, b]$ be a closed, bounded interval and let $f : I \to \mathbb{R}$ be on I. If $k \in \mathbb{R}$ is any number satisfying

$$\inf f(I) \leq k \leq \sup f(I),$$

then there exists a number $c \in I$ such that $f(c) = k$.

Theorem 5.3.5. Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then the set $f(I) := \{f(x) : x \in I\}$ is a closed bounded interval.

Theorem 5.3.6 (**Preservation of Intervals Theorem**). Let I be an interval and let $f : I \to \mathbb{R}$ be continuous on I. Then the set $f(I)$ is an interval.

5.4 Uniform Continuity

Definition 5.4.1. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. We say that f is **uniformly continuous** on A if for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that if $x, u \in A$ are any numbers satisfying $|x - u| < \delta(\varepsilon)$, then $|f(x) - f(u)| < \varepsilon$.

Theorem 5.4.1 (**Nonuniform Continuity Criteria**). Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Then the following statements are equivalent:

1. f is not uniformly continuous on A.
2. There exists an $\varepsilon_0 > 0$ such that for every $\delta > 0$ there are points x_δ, u_δ in A such that $|x_\delta - u_\delta| < \delta$ and $|f(x_\delta) - f(u_\delta)| \geq \varepsilon_0$.
3. There exists an $\varepsilon_0 > 0$ and two sequences (x_n) and (u_n) in A such that $\lim(x_n - u_n) = 0$ and $|f(x_n) - f(u_n)| \geq \varepsilon_0 = 1$ for all $n \in \mathbb{N}$.

Theorem 5.4.2 (**Uniform Continuity Theorem**). Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I.

Definition 5.4.2. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. If there exists a constant $K > 0$ such that

$$(4) \qquad |f(x) - f(u)| \leq K|x - u|$$

for all $x, u \in A$, then f is said to be a **Lipschitz function** (or to satisfy a **Lipschitz condition**) on A.

The condition (4) that a function $f : I \to \mathbb{R}$ on an interval I is a Lipschitz function can be interpreted geometrically as follows. If we write the condition as

$$\left|\frac{f(x)-f(u)}{x-u}\right| \le K,\ x,u \in I,\ x \neq u,$$

then the quantity inside the absolute values is the slope of a line segment joining the points $(x, f(x))$ and $(u, f(u))$. Thus a function f satisfies a Lipschitz condition if and only if the slopes of all line segments joining two points on the graph of $y = f(x)$ over I are bounded by some number K.

Theorem 5.4.3. If $f : A \to \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on A.

Theorem 5.4.4. If $f : A \to \mathbb{R}$ is uniformly continuous on a subset A of $\mathbb{R}$ and if (x_n) is a Cauchy sequence in A, then $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$.

Theorem 5.4.5 (Continuous Extension Theorem). A function f is uniformly continuous on the interval (a, b) if and only if it can be defined at the endpoints a and b such that the extended function is continuous on $[a, b]$.

Definition 5.4.3. A function $s : [a, b] \to \mathbb{R}$ is called a **step function** if $[a, b]$ is the union of a finite number of nonoverlapping intervals $I_1, I_2, \ldots, I_n$ such that s is constant on each interval, that is, $s(x) = c_k$ for all $x \in I_k,\ k = 1, 2, \ldots, n$.

Theorem 5.4.6. Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. If $\varepsilon > 0$, then there exists a step function $s_\varepsilon : I \to \mathbb{R}$ such that $|f(x) - s_\varepsilon(x)| < \varepsilon$ for all $x \in I$.

Corollary 5.4.1. Let $I := [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. If $\varepsilon > 0$, there exists a natural number m such that if we divide I into m disjoint intervals I_k having length $h := (b - a/m)$, then the step function s_ε defined in equation (5) satisfies $|f(x) - s_\varepsilon(x)| < \varepsilon$ for all $x \in I$.

Definition 5.4.4. Let $I := [a, b]$ be an interval. Then a function $g : I \to \mathbb{R}$ is said to be **piecewise linear** on I if I is the union of a finite number of disjoint intervals $I_1, \ldots, I_m$, such that the restriction of g to each interval I_k is a linear function.

Theorem 5.4.7. Let I be a closed bounded interval and let $: I \to \mathbb{R}$ be continuous on I. If $\varepsilon > 0$, then there exists a continuous piecewise linear function $g_\varepsilon : I \to \mathbb{R}$ such that $|f(x) - g_\varepsilon(x)| < \varepsilon$ for all $x \in I$.

Theorem 5.4.8 (Wierstrass Approximation Theorem). Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be a continuous function. If $\varepsilon > 0$ is given, then there exists a polynomial function p_ε such that $|f(x) - p_\varepsilon(x)| < \varepsilon$ for all $x \in I$.

5.5 Continuity and Gauges

Definition 5.5.1. A **partition** of an interval $I := [a, b]$ is a collection $\mathbb{P} = \{I_1, \ldots, I_n\}$ of non-overlapping closed intervals whose union is $[a, b]$. We ordinarily denote the intervals by $I_i := [x_{i-1}, x_i]$, where

$$a = x_0 < \cdots < x_{i-1} < x_i < \cdots < x_n = b.$$

The points x_i $(i - 0, \ldots, n)$ are called the **partition points** of $\mathbb{P}$. If a point t_i has been chosen from each interval I_i for $i = 1, \ldots, n$, then the points t_i are called the **tags** and the set of ordered pairs

$$\mathbb{P} = \{(I_1, t_1), \ldots, (I_n, t_n)\}$$

is called a **tagged partition** of I. (The dot signifies that the partition is tagged.)

Definition 5.5.2. A **gauge** on I is a strictly positive function defined on I. If δ is a gauge on I, then a (tagged) partition $\mathbb{P}$ is said to be δ-**fine** if

$$t_i \in I_i \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)], \qquad \text{for} \qquad i = 1, \ldots, n.$$

We note that the notion of δ-fineness requires that the partition be tagged, so we do not need to say "tagged partition" in this case.

Lemma 5.5.1. If a partition $\mathbb{P}$ of $I := [a, b]$ is δ-fine and $x \in I$, then there exists a tag t_i in $\mathbb{P}$ such that $|x - t_i| \leq \delta(t_i)$.

Theorem 5.5.1. If δ is a gauge defined on the interval $[a, b]$, then there exists a δ-fine partition of $[a, b]$.

5.6 Monotone and Inverse Functions

Definition 5.6.1. Recall that if $A \subseteq \mathbb{R}$, then a function $f : A \to \mathbb{R}$ is said to be **increasing on** A if whenever $x_1, x_2 \in A$ and $x_1 \leq x_2$, then $f(x_1) \leq f(x_2)$. The function f is said to be **strictly increasing on** A if whenever $x_1, x_2 \in A$ and $x_1 < x_2$, then $f(x_1) < f(x_2)$. Similarly, $g : A \to \mathbb{R}$ is said to be **decreasing on** A if whenever $x_1, x_2 \in A$ and $x_1 \leq x_2$ then $g(x_1) \geq g(x_2)$. The function g is said to be **strictly decreasing on** A if whenever $x_1, x_2 \in A$ and $x_1 < x_2$ then $g(x_1) > g(x_2)$. If a function is either increasing or decreasing on A, we say that it is **monotone** on A. If f is either strictly increasing or strictly decreasing on A, we say that f is **strictly monotone** on A.

Theorem 5.6.1. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be increasing on I. Suppose that $c \in I$ is not an endpoint of I. Then

1. $\lim\limits_{x \to c^-} f = \sup\{f(x) : x \in I,\ x < c\}$,

2. $\lim\limits_{x \to c^+} f = \inf\{f(x) : x \in I,\ x > c\}$.

Corollary 5.6.1. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be increasing on I. Suppose that $c \in I$ is not an endpoint of I. Then the following statements are equivalent.

1. f is continuous at c.

2. $\lim\limits_{x \to c^-} f = f(c) = \lim\limits_{x \to c^+}$.

3. $\sup\{f(x) : x \in I,\ x < c\} = f(c) = \inf\{f(x) : x \in I,\ x > c\}$.

Definition 5.6.2. If $f : I \to \mathbb{R}$ is increasing on I and if c is not an endpoint of I, we define the **jump of** f **at** c to be $j_f(c) := \lim\limits_{x \to c^+} f - \lim\limits_{x \to c^-} f$. It follows from *Theorem 5.6.1* that

$$j_f(c) = \inf\{f(x) \ : \ x \in I, \ x > c\} - \sup\{f(x) \ : \ x \in I, \ x < c\}$$

for an increasing function. If the left endpoint a of I belongs to I, we define the **jump of** f **at** a to be $j_f(a) := \lim\limits_{x \to a^+} f - f(a)$. If the right endpoint b belongs to I, we define the **jump of** f **at** b to be $j_f(b) := f(b) - \lim\limits_{x \to b^-} f$.

Theorem 5.6.2. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be increasing on I. If $c \in I$, then f is continuous at c if and only if $j_f(c) = 0$.

Theorem 5.6.3. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be monotone on I. Then the set of points $D \subseteq I$ at which f is discontinuous is a countable set.

Theorem 5.6.4 (**Continuous Inverse Theorem**)**.** Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be strictly monotone and continuous on I. Then the function g inverse to f is strictly monotone and continuous on $J := f(I)$.

Definition 5.6.3.

1. If $m, n \in \mathbb{N}$ and $x \geq 0$, we define $x^{m/n} := (x^{1/n})^m$.
2. If $m, n \in N$ and $x > 0$, we define $x^{-m/n} := (x^{1/n})^{-m}$.

Theorem 5.6.5. If $m \in \mathbb{Z}, \ n \in \mathbb{N}$, and $x > 0$, then $x^{m/n} = (x^m)^{1/n}$.

Chapter 6

Differentiation

6.1 The Derivative

Definition 6.1.1. Let $I \subseteq \mathbb{R}$ be an interval, let $f : I \to \mathbb{R}$, and let $c \in I$. We say that a real number L is the **derivative of** f **at** c if given any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $x \in I$ satisfies $0 < |x - c| < \delta(\varepsilon)$, then

$$\left| \frac{f(x) - f(c)}{x - c} - L \right| < \varepsilon.$$

In this case we say that f is **differentiable** at c, and we write $f'(c)$ for L. In other words, the derivative of f at c is given by the limit

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

provided this limit exists. (We allow the possibility that c may be the endpoint of the interval.)

Theorem 6.1.1. If $f : I \to \mathbb{R}$ has a derivative at $c \in I$, then f is continuous at c.

Theorem 6.1.2. Let $I \subseteq \mathbb{R}$ be an interval, let $c \in I$, and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be functions that are differentiable at c. Then:

1. If $\alpha \in \mathbb{R}$, then the function αf is differentiable at c, and

$$(\alpha f)'(c) = \alpha f'(c)$$

2. The function $f + g$ is differentiable at c, and

$$(f + g)'(c) = f'(c) + g'(c)$$

3. (Product Rule) The function fg is differentiable at c, and

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

4. (Quotient Rule) If $g(c) \neq 0$, then the function f/g is differentiable at c, and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$$

Corollary 6.1.1. If $f_1, f_2, \ldots, f_n$ are functions on an interval I to $\mathbb{R}$ that are differentiable at $c \in I$, then:

1. The function $f_1 + f_2 + \cdots + f_n$ is differentiable at c and

$$(f_1 + f_2 + \cdots + f_n)'(c) = f_1'(c) + f_2'(c) + \cdots + f_n'(c)$$

2. The function $f_1 f_2 \ldots f_n$ is differentiable at c and

$$(f_1 f_2 \ldots f_n)'(c) = f_1'(c) f_2(c) \ldots f_n(c) + f_1(c) f_2'(c) \ldots f_n(c) + \cdots + f_1(c) f_2(c) \ldots f_n'(c).$$

An important special case of the extended product rule occurs if the functions are equal, that is, $f_1 = f_2 = \cdots = f_n = f$. Then the above becomes

$$(f^n)'(c) = n(f(c))^{n-1} f'(c)$$

Theorem 6.1.3 (Carathéodory's Theorem). Let f be defined on an interval I containing the point c. Then f is differentiable at c if and only if there exists a function φ on I that is continuous at c and satisfies

$$f(x) - f(c) = \varphi(x)(x - c) \quad \text{for} \quad x \in I$$

In this case, we have $\varphi(c) = f'(c)$.

Theorem 6.1.4 (Chain Rule). Let I, J be intervals in $\mathbb{R}$, let $g : I \to \mathbb{R}$ and $f : J \to \mathbb{R}$ be functions such that $f(J) \subseteq I$, and let $c \in J$. If f is differentiable at c and if g is differentiable at $f(c)$, then the composite function $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c).$$

Theorem 6.1.5. Let I be an interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be strictly monotone and continuous on I. Let $J := f(I)$ and let $g : J \to \mathbb{R}$ be the strictly monotone and continuous function inverse to f. If f is differentiable at $c \in I$ and $f'(c) \neq 0$, then g is differentiable at $d := f(c)$ and

$$g'(d) = \frac{1}{f'(c)} = \frac{1}{f'(g(d))}$$

Theorem 6.1.6. Let I be an interval and let $f : I \to \mathbb{R}$ be strictly monotone on I. Let $J := f(I)$ and let $g : J \to \mathbb{R}$ be the function inverse to f. If f is differentiable on I and $f'(x) \neq 0$ for $x \in I$, then g is differentiable on J and

$$g' = \frac{1}{f' \circ g}$$

6.2 The Mean Value Theorem

Theorem 6.2.1 (Interior Extremum Theorem). Let c be an interior point of of the interval I at which $f : I \to \mathbb{R}$ has a relative extremum. If the derivative of f at c exists, then $f'(c) = 0$.

Corollary 6.2.1. Let $f : I \to \mathbb{R}$ be continuous on an interval I and suppose that f has a relative extremum at an interior point c of I. Then either the derivative of f at c does not exist, or it is equal to zero.

Theorem 6.2.2 (Rolle's Theorem). Suppose that f is continuous on a closed interval $I := [a, b]$, that the derivative f' exists at every point of the open interval (a, b), and that $f(a) = f(b) = 0$. Then there exists at least one point c in (a, b) such that $f'(c) = 0$.

Theorem 6.2.3 (Mean Value Theorem). Suppose that f is continuous on a closed interval $I := [a, b]$, and that f has a derivative in the open interval (a, b). Then there exists at least one point c in (a, b) such that

$$f(b) - f(a) = f'(c)(b - a)$$

Theorem 6.2.4. Suppose that f is continuous on the closed interval $I := [a, b]$, that f is differentiable on the open interval (a, b), and that $f'(x) = 0$ for $x \in (a, b)$. Then f is constant on I.

Corollary 6.2.2. Suppose that f and g are continuous on $I := [a, b]$, that they are differentiable on (a, b), and that $f'(x) = g'(x)$ for all $x \in (a, b)$. Then there exists a constant C such that $f = g + C$ on I.

Theorem 6.2.5. Let $f : I \to \mathbb{R}$ be differentiable on the interval I. Then:

1. f is increasing on I if and only if $f'(x) \geq 0$ for all $x \in I$.

2. f is decreasing on I if and only if $f'(x) \leq 0$ for all $x \in I$.

Theorem 6.2.6 (First Derivative Test for Extrema). Let f be continuous on the interval $I := [a, b]$ and let c be an interior point of I. Assume that f is differentiable on (a, c) and (c, b). Then:

1. If there is a neighborhood $(c - \delta, c + \delta) \subseteq I$ such that $f'(x) \geq 0$ for $c - \delta < x < c$ and $f'(x) \leq 0$ for $c < x < c + \delta$, then f has a relative maximum at c.

2. If there is a neighborhood $(c - \delta, c + \delta) \subseteq I$ such that $f'(x) \leq 0$ for $c - \delta < x < c$ and $f'(x) \geq 0$ for $c < x < c + \delta$, then f has a relative maximum at c.

Lemma 6.2.1. Let $I \subseteq \mathbb{R}$ be an interval, let $f : I \to \mathbb{R}$, let $c \in I$, and assume that f has a derivative at c. Then:

1. If $f'(c) > 0$, then there is a number $\delta > 0$ such that $f(x) > f(c)$ for $x \in I$ such that $c < x < c + \delta$.

2. If $f'(c) < 0$, then there is a number $\delta > 0$ such that $f(x) > f(c)$ for $x \in I$ such that $c - \delta < x < c$.

Theorem 6.2.7 (Darboux's Theorem). If f is differentiable on $I = [a, b]$ and if k is a number between $f'(a)$ and $f'(b)$, then there is at least one point c in (a, b) such that $f'(c) = k$.

6.3 L'Hopital's Rules

Theorem 6.3.1. Let f and g be defined on $[a, b]$, let $f(a) = g(a) = 0$, and let $g(x) \neq 0$ for $a < x < b$. If f and g are differentiable at a and if $g'(a) \neq 0$, then the limit of f/g at a exists and is equal to $f'(a)/g'(a)$. Thus

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Theorem 6.3.2 (Cauchy Mean Value Theorem). Let f and g be continuous on $[a, b]$ and differentiable on (a, b), and assume that $g'(x) \neq 0$ for all x in (a, b). Then there exists c in (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Theorem 6.3.3 (L'Hopital's Rule, I). Let $-\infty \leq a < b \leq \infty$ and let f, g be differentiable on (a, b) such that $g'(x) \neq 0$ for all $x \in (a, b)$. Suppose that

$$\lim_{x \to a^+} f(x) = 0 = \lim_{x \to a^+} g(x)$$

1. If $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$, then $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$.

2. If $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \in \{-\infty, \infty\}$, then $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$.

Theorem 6.3.4 (L'Hopital's Rule, II). Let $-\infty \leq a < b \leq \infty$ and let f, g be differentiable on (a, b) such that $g'(x) \neq 0$ for all $x \in (a, b)$. Suppose that

$$\lim_{x \to a^+} g(x) = \pm\infty$$

1. If $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$, then $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$.

2. If $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \in \{-\infty, \infty\}$, then $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$.

6.4 Taylor's Theorem

Theorem 6.4.1 (Taylor's Theorem). Let $n \in \mathbb{N}$, let $I := [a, b]$, and let $f : I \to \mathbb{R}$ be such that f and its derivative $f', f'', \ldots, f^{(n)}$ are continuous on I and that $f^{(n+1)}$ exists on (a, b). If $x_0 \in I$, then for any x in I there exists a point c between x and x_0 such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2$$

$$+ \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

Theorem 6.4.2. Let I be an interval, let x_0 be an interior point of I, and let $n \geq 2$. Suppose that the derivatives $f', f'', \ldots, f^{(n)}$ exist and are continuous in a neighborhood of x_0 and that $f'(x_0) = \cdots = f^{(n-1)(x_0)}$, but $f^{(n)}(x_0) \neq 0$.

1. If n is even and $f^{(n)}(x_0) > 0$, then f has a relative minimum at x_0.

2. If n is even and $f^{(n)} < 0$, then f has a relative maximum at x_0.

3. If n is odd, then f has neither a relative minimum nor relative maximum at x_0.

Definition 6.4.1. Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is said to be **convex** on I if for any t satisfying $0 \le t \le 1$ and any points x_1, x_2 in I, we have

$$f((1-t)x_1 + tx_2) \le (1-t)f(x_1) + tf(x_2).$$

Theorem 6.4.3. Let I be an open interval and let $f : I \to \mathbb{R}$ have a second derivative on I. Then f is a convex function on I if and only if $f''(x) \ge 0$ for all $x \in I$.

Theorem 6.4.4 (**Newton's Method**)**.** Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ be twice differentiable on I. Suppose that $f(a)f(b) < 0$ and that there are constants m, M such that $|f'(x)| \ge m > 0$ and $|f''(x)| \le M$ for $x \in I$ and let $K := M/2m$. Then there exists a subinterval I^* containing a zero r of f such that for any $x_1 \in I^*$ the sequence (x_n) defined by

$$|x_{n+1} - r| \le x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{for all} \quad n \in \mathbb{N},$$

belongs to I^* and (x_n) converges to r. Moreover

$$|x_{n+1} - r| \le K|x_n - r|^2 \quad \text{for all} \quad n \in \mathbb{N}.$$

Chapter 7

The Riemann Integral

7.1 Riemann Integral

Definition 7.1.1. If $I := [a, b]$ is a closed bounded interval in $\mathbb{R}$, then a **partition** of I is a finite, ordered set $\mathcal{P} := (x_0, x_1, \ldots, x_{n-1}, x_n)$ of point in I such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

Often we will denote the partition $\mathcal{P}$ by the notation $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^n$. We define the **norm** (or **mesh**) of $\mathcal{P}$ to be the number

$$||\mathcal{P}|| := \max\{x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}\}$$

Thus the norm of a partition is merely the length of the largest subinterval into which the partition divides $[a, b]$. Clearly, many partitions have the same norm, so the partition is *not* a function of the norm.
If a point t_i has been selected from each subinterval $I_i = [x_{i-1}, x_i]$, for $i = 1, 2, \ldots, n$, then the points are called **tags** of the subintervals of I_i. A set of ordered pairs

$$\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$$

of subintervals and corresponding tags is called a **tagged partition** of I.
If $\dot{\mathcal{P}}$ is the tagged partition given above, we define the **Riemann sum** of a function $f : [a, b] \to \mathbb{R}$ corresponding to $\dot{\mathcal{P}}$ to be the number

$$S(f; \dot{\mathcal{P}}) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

We will also use this notation when $\dot{\mathcal{P}}$ denotes a *subset* of a partition, and not the entire partition.

Definition 7.1.2. A function $f : [a, b] \to \mathbb{R}$ is said to be **Riemann integrable** on $[a, b]$ if there exists a number $L \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ is any tagged partition of $[a, b]$ with $||\dot{\mathcal{P}}|| < \delta_\varepsilon$, then

$$|S(f; \dot{\mathcal{P}}) - L| < \varepsilon$$

The set of all Riemann integrable functions on $[a, b]$ will be denoted by $\mathcal{R}[a, b]$.

Remark 7.1.1. It is sometimes said that the integral L is "the limit" of the Riemann sums $S(f : \dot{\mathcal{P}})$ as the norm $||\dot{\mathcal{P}}|| \to 0$. However, since $S(f; \dot{\mathcal{P}})$ is not a function of $||\dot{\mathcal{P}}||$, this **limit** is not of the type that we have studied before.

First we will show that if $f \in \mathcal{R}[a, b]$, then the number L is uniquely determined. It will be called the **Riemann integral of** f over $[a, b]$. Instead of L, we will usually write

$$L = \int_a^b f \quad \text{or} \quad \int_a^b f(x)dx$$

Theorem 7.1.1. If $f \in \mathcal{R}[a, b]$, then the value of the integral is uniquely determined.

Theorem 7.1.2. If g is Riemann integrable on $[a, b]$ and if $f(x) = g(x)$ except for a finite number of points in $[a, b]$, then f is Riemann integrable and $\displaystyle\int_a^b f = \int_b^a g$.

Theorem 7.1.3. Suppose that f and g are in $\mathcal{R}[a, b]$. Then:

1. If $k \in \mathbb{R}$, the function kf is in $\mathcal{R}[a, b]$ and

$$\int_a^b kf = k \int_a^b f$$

2. The function $f + g$ is in $\mathcal{R}[a, b]$ and

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g$$

3. If $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f \leq \int_a^b g$$

Theorem 7.1.4. If $f \in \mathcal{R}[a, b]$, then f is bounded on $[a, b]$.

7.2 Riemann Integrable Functions

Theorem 7.2.1 (Cauchy Criterion). A function: $[a, b] \to \mathbb{R}$ belongs to $\mathcal{R}[a, b]$ if and only if for every $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ are any tagged partitions of $[a, b]$ with $||\dot{\mathcal{P}}|| < \eta_\varepsilon$ and $||\dot{\mathcal{Q}}|| < \eta_\varepsilon$, then

$$|S(f; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{Q}})| < \varepsilon$$

Theorem 7.2.2 (Squeeze Theorem). Let $f : [a, b] \to \mathbb{R}$. Then $f \in \mathcal{R}[a, b]$ if and only if for every $\varepsilon > 0$ there exist functions α_ε and ω_ε in $\mathcal{R}[a, b]$ with

$$\alpha_\varepsilon(x) \leq f(x) \leq \omega_\varepsilon(x) \ \forall \ x \in [a, b]$$

and such that

$$\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon$$

Lemma 7.2.1. If J is a subinterval of $[a,b]$ having endpoints $c < d$ and if $\varphi_J(x) := 1$ for $x \in J$ and $\varphi_J(x) := 0$ elsewhere in $[a,b]$, then $\varphi_J \in \mathcal{R}[a,b]$ and $\displaystyle\int_a^b \varphi_J = d - c$.

Theorem 7.2.3. If $\varphi : [a,b] \to \mathbb{R}$ is a step function, then $\varphi \in \mathcal{R}[a,b]$.

Theorem 7.2.4. If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then $f \in \mathcal{R}[a,b]$.

Theorem 7.2.5. If $f : [a,b] \to \mathbb{R}$ is monotone on $[a,b]$, then $f \in \mathcal{R}[a,b]$.

Theorem 7.2.6 (Additivity Theorem). Let $f := [a,b] \to \mathbb{R}$ and let $c \in (a,b)$. Then $f \in \mathcal{R}[a,b]$ if and only if its restrictions to $[a,c]$ and $[c,b]$ are both Riemann integrable. In this case

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Corollary 7.2.1. If $f \in \mathcal{R}[a,b]$, and if $[c,d] \subseteq [a,b]$, then the restriction of f to $[c,d]$ is in $\mathcal{R}[c,d]$.

Corollary 7.2.2. If $f \in \mathcal{R}[a,b]$ and if $a = c_0 < c_1 < \cdots < c_m = b$, then the restrictions of f to each of the subintervals $[c_{i-1}, c_i]$ are Riemann integrable and

$$\int_a^b f = \sum_{i=1}^{m} \int_{c_{i-1}}^{c_i} f$$

Definition 7.2.1. If $f \in \mathcal{R}[a,b]$ and if $\alpha, \beta \in [a,b]$ with $\alpha < \beta$, we define

$$\int_\beta^\alpha f := -\int_\alpha^\beta f \quad \text{and} \quad \int_\alpha^\alpha f := 0$$

Theorem 7.2.7. If $f \in \mathcal{R}[a,b]$ and if α, β, γ are any numbers in $[a,b]$, then

$$\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f$$

in the sense that the existence of any two of these integrals implies the existence of the third integral and the equality.

7.3 The Fundamental Theorem

Theorem 7.3.1 (Fundamental Theorem of Calculus (First Form)). Suppose there is a **finite** set E in $[a,b]$ and functions $f, F := [a,b] \to \mathbb{R}$ such that

1. F is continuous on $[a,b]$,
2. $F'(x) = f(x)$ for all $x \in [a,b] \setminus E$,
3. f belongs to $\mathcal{R}[a,b]$.

Then we have

$$\int_a^b f = F(b) - F(a)$$

Remark 7.3.1. If the function F is differentiable at every point of $[a, b]$, then (by *Theorem 6.1.2*) hypothesis (a) is automatically satisfied. If f is not defined for some point $c \in E$, we take $f(c) := 0$. Even if F is differentiable at every point of $[a, b]$, condition (c) is *not automatically satisfied*, since there exists functions F such that F' is not Riemann integrable.

Definition 7.3.1. If $f \in \mathcal{R}[a, b]$, then the function defined by

$$F(z) := \int_a^z f \text{ for } z \in [a, b]$$

is called the **indefinite integral** of f with **basepoint** a. (sometimes a point other than a is used as a basepoint)

Theorem 7.3.2. The indefinite integral F defined by the above definition is continuous on $[a, b]$. In fact, if $|f(x)| \leq M$ for all $x \in [a, b]$, then $|F(z) - F(w)| \leq M|z - w|$ for all $z, w \in [a, b]$.

Theorem 7.3.3 (Fundamental Theorem of Calculus (Second Form)). Let $f \in \mathcal{R}[a, b]$ and let f be continuous at a point $c \in [a, b]$. Then the indefinite integral, defined by *Definition 7.3.1*, is differentiable at c and $F'(c) = f(c)$.

Theorem 7.3.4. If f is continuous on $[a, b]$, then the indefinite integral F, defined by *Definition 7.3.1*, is differentiable on $[a, b]$, and $F'(x) = f(x)$ for all $x \in [a, b]$.

Theorem 7.3.5 (Substitution Theorem). Let $J := [\alpha, \beta]$ and let $\varphi : J \to \mathbb{R}$ have a continuous derivative on J. If $f : I \to \mathbb{R}$ is continuous on an interval I containing $\varphi(J)$, then

$$\int_\alpha^\beta f(\varphi(t)) \cdot \varphi'(t) dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x) dx$$

Definition 7.3.2.

1. A set $Z \subset \mathbb{R}$ is said to be a **null set** if for every $\varepsilon > 0$ there exists a countable collection $\{(a_k, b_k)\}_{k=1}^{\infty}$ of open intervals such that

$$Z \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k) \text{ and } \sum_{k=1}^{\infty} (b_k - a_k) \leq \varepsilon$$

2. If $Q(x)$ is a statement about the point $x \in I$, we say that $Q(x)$ holds **almost everywhere** on I (or for **almost every** $x \in I$), if there exists a null set $Z \subset I$ such that $Q(x)$ holds for all $x \in I \setminus Z$. In this case, we may write

$$Q(x) \text{ for a.e. (almost everywhere) } x \in I$$

Theorem 7.3.6 (Lebesgue's Integrability Criterion). A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if it is continuous almost everywhere on $[a, b]$.

Theorem 7.3.7 (**Composition Theorem**). Let $f \in \mathcal{R}[a,b]$ with $f([a,b]) \subseteq [c,d]$ and let $\varphi : [c,d] \to \mathbb{R}$ be continuous. Then the composition $\varphi \circ f$ belongs to $\mathcal{R}[a,b]$.

Corollary 7.3.1. Suppose that $f \in \mathcal{R}[a,b]$. Then its absolute value $|f|$ is in $\mathcal{R}[a,b]$, and

$$\left|\int_a^b f\right| \le \int_a^b |f| \le M(b-a),$$

where $|f(x)| \le M$ for all $x \in [a,b]$.

Theorem 7.3.8 (**The Product Theorem**). If f and g belong to $\mathcal{R}[a,b]$, then the product fg belongs to $\mathcal{R}[a,b]$.

Theorem 7.3.9 (**Integration by Parts**). Let F, G be differentiable on $[a,b]$ and let $f := F'$ and $g := G'$ belong to $\mathcal{R}[a,b]$. Then

$$\int_a^b fG = FG \Big|_a^b - \int_a^b Fg$$

Theorem 7.3.10 (**Taylor's Theorem with the Remainder**). Suppose that $f', \dots, f^{(n)}, f^{(n+1)}$ exist on $[a,b]$ and that $f^{(n+1)} \in \mathcal{R}[a,b]$. Then we have

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + R_n,$$

where the remainder is given by

$$R_n = \frac{1}{n!}\int_a^b f^{(n+1)}(t) \cdot (b-t)^n dt$$

7.4 The Darboux Integral

Definition 7.4.1 (**Upper and Lower Sums**). Let $f : I \to \mathbb{R}$ be a bounded function on $I = [a,b]$ and let $\mathcal{P} = (x_0, x_1, \dots, x_n)$ be a partition of I. for $k = 1, 2, \dots, n$ we let

$$m_k := \inf\{f(x) : x \in [x_{k-1}, x_k]\}, \qquad M_k := \sup\{f(x) : x \in [x_{k-1}, x_k]\}$$

The **lower sum** of f corresponding to the partition $\mathcal{P}$ is defined to be

$$L(f;\mathcal{P}) := \sum_{k=1}^{n} m_k(x_k - x_{k-1})$$

and the **upper sum** of f corresponding to $\mathcal{P}$ is defined to be

$$U(f;\mathcal{P}) := \sum_{k=1}^{n} M_k(x_k - x_{k-1})$$

Lemma 7.4.1. If $f := I \to \mathbb{R}$ is bounded and $\mathcal{P}$ is any partition of I, then $L(f;\mathcal{P}) \le U(f;\mathcal{P})$.

Definition 7.4.2. If $\mathcal{P} := (x_0, x_1, \ldots, x_n)$ and $\mathcal{Q} := (y_0, y_1, \ldots, y_m)$ are partitions of I, we say that $\mathcal{Q}$ is a **refinement of** $\mathcal{P}$ if each partition point $x_k \in \mathcal{P}$ also belongs to $\mathcal{Q}$ (that is, if $\mathcal{P} \subseteq \mathcal{Q}$). A refinement $\mathcal{Q}$ of a partition $\mathcal{P}$ can be obtained by adjoining a finite number of points to $\mathcal{P}$.

Lemma 7.4.2. If $f : I \to \mathbb{R}$ is bounded, if $\mathcal{P}$ is a partition of I, and if Q is a refinement of $\mathcal{P}$, then

$$L(f;\mathcal{P}) \leq L(f;\mathcal{Q}) \text{ and } U(f;\mathcal{Q}) \leq U(f;\mathcal{P})$$

Lemma 7.4.3. Let $f : I \to \mathbb{R}$ be bounded. If $\mathcal{P}_1, \mathcal{P}_2$ are any two partitions of I, then $L(f;\mathcal{P}_1) \leq U(f;\mathcal{P}_2)$.

Definition 7.4.3. We shall denote the collection of all partitions of the interval I by $\mathscr{P}(I)$. Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ be a bounded function. The **lower integral of** f **on** I is the number

$$L(f) := \sup\{L(f;\mathcal{P}) : \mathcal{P} \in \mathscr{P}(I)\}$$

and the **upper integral of** f **on** I is the number

$$U(f) := \inf\{U(f;\mathcal{P}) : \mathcal{P} \in \mathscr{P}(I)\}$$

Theorem 7.4.1. Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be a bounded function. Then the lower integral $L(f)$ and the upper integral $U(f)$ of f on I exist. Moreover,

$$L(f) \leq U(f)$$

Definition 7.4.4. Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be a bounded function. Then f is said to be **Darboux integrable on** I if $L(f) = U(f)$. In this case the **Darboux Integral of** f **over** I is defined to be the value $L(f) = U(f)$.

Theorem 7.4.2 (Integrability Criterion). Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ be a bounded function on I. Then f is Darboux integrable on I if and only if for each $\varepsilon > 0$ there is a partition $\mathcal{P}_\varepsilon$ of I such that

$$U(f;\mathcal{P}_\varepsilon) - L(f;\mathcal{P}_\varepsilon) < \varepsilon$$

Corollary 7.4.1. Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be a bounded function. If $\{P_n : n \in \mathbb{N}\}$ is a sequence of partitions on I such that

$$\lim_n (U(f;P_n) - L(f;P_n)) = 0,$$

then f is integrable and $\lim_n L(f;P_n) = \int_a^b f = \lim_n U(f;P_n)$.

Theorem 7.4.3. If the function f on the interval $I = [a, b]$ is either continuous or monotone on I, then f is Darboux integrable on I.

Theorem 7.4.4 (Equivalence Theorem). A function f on $I = [a, b]$ is Darboux integrable if and only if it is Riemann integrable.

7.5 Approximate Integration

Equal Partitions If $f : [a,b] \to \mathbb{R}$ is continuous, we know that its Riemann integral exists. To find an approximate value for this integral with the minimum amount of calculation, it is convenient to consider partitions $\mathcal{P}_n$ of $[a,b]$ into n *equal* subintervals having length $h_n := (b-a)/n$. Hence $\mathcal{P}_n$ is the partition:

$$a < a + h_n < a + 2h_n < \cdots < a + nh_n = b$$

If we pick our tag points to be the *left endpoints* and the *right endpoints* of the subintervals, we obtain the n**th left approximation** given by

$$L_n(f) := h_n \sum_{k=1}^{n-1} f(a + kh_n)$$

and the n**th right approximation** given by

$$R_n(f) := h_n \sum_{k=1}^{n} f(a + kh_n)$$

Theorem 7.5.1. If $f : [a,b] \to \mathbb{R}$ is monotone and if $T_n(f)$ is given by

$$T_n(f) := h_n \left(\frac{1}{2} f(a) + \sum_{k=1}^{n-1} f(a + kh_n) + \frac{1}{2} f(b) \right)$$

then

$$\left| \int_a^b f - T_n(f) \right| \leq |f(b) - f(a)| \cdot \frac{(b-a)}{2n}$$

Note that the function $T_n(f)$ defined above is called the n**th Trapezoidal Approximation** of f

Theorem 7.5.2. Let f, f', and f'' be continuous on $[a,b]$ and let $T_n(f)$ be the nth Trapezoidal Approximation. Then there exists $c \in [a,b]$ such that

$$T_n(f) - \int_a^b f = \frac{(b-a)h_n^2}{12} \cdot f''(c)$$

Corollary 7.5.1. Let f, f', and f'' be continuous, and let $|f''(x)| \leq B_2$ for all $x \in [a,b]$. Then

$$\left| T_n(f) - \int_a^b f \right| \leq \frac{(b-a)h_n^2}{12} \cdot B_2 = \frac{(b-a)^3}{12n^2} \cdot B_2$$

If $\mathcal{P}_n$ is the equally spaced partition given before, the **Midpoint Approximation** of f is given by

$$M_n(f) := h_n \left(f\left(a + \frac{1}{2}h_n\right) + f\left(a + \frac{3}{2}h_n\right) + \cdots + f\left(a\left(n - \frac{1}{2}\right)h_n\right)\right) = h_n \sum_{k=1}^{n} f\left(a + \left(k - \frac{1}{2}\right)h_n\right)$$

Theorem 7.5.3. Let f, f', and f'' be continuous on $[a, b]$ and let $M_n(f)$ be the nth Midpoint Approximation. Then there exists $\gamma \in [a, b]$ such that

$$\int_a^b f - M_n(f) = \frac{(b-a)h_n^2}{24} \cdot f''(\gamma)$$

Corollary 7.5.2. Let f, f', and f'' be continuous, and let $|f''(x)| \leq B_2$ for all $x \in [a, b]$. Then

$$\left| M_n(f) - \int_a^b f \right| \leq \frac{(b-a)h_n^2}{24} \cdot B_2 = \frac{(b-a)^3}{24n^2} \cdot B_2$$

The n**th Simpson Approximation** is defined by

$$S_n(f) := \frac{1}{3}h_n(f(a)+4f(a+h_n)+2f(a+2h_n)+4f(a+3h_n)+2f(a+4h_n)+\cdots+2f(b-2h_n)+4f(b-h_n)+f(b))$$

Theorem 7.5.4. Let $f, f', f'', f^{(3)}$, and $f^{(4)}$ be continuous on $[a, b]$ and let $n \in \mathbb{N}$ be even. If $S_n(f)$ is the nth Simpson Approximation, then there exists $c \in [a, b]$ such that

$$S_n(f) - \int_a^b f = \frac{(b-a)h_n^4}{180} \cdot f^{(4)}(c)$$

Corollary 7.5.3. Let $f, f', f'', f^{(3)}$, and $f^{(4)}$ be continuous on $[a, b]$ and let $|f^{(4)}| \leq B_4$ for all $x \in [a, b]$. Then

$$\left| S_n(f) - \int_a^b f \right| \leq \frac{(b-a)h_n^4}{180} \cdot B_4 = \frac{(b-a)^5}{180n^4} \cdot B_4$$

Remark 7.5.1. The nth Midpoint Approximation $M_n(f)$ can be used to "step up" to the $(2n)$th Trapezoidal and Simpson Approximations by using the formulas

$$T_{2n}(f) = \frac{1}{2}M_n(f) + \frac{1}{2}T_n(f) \quad \text{and} \quad S_{2n}(f) = \frac{2}{3}M_n(f) + \frac{1}{3}T_n(f)$$

that are given in the exercises. Thus once the initial Trapezoidal Approximation $T_1 = T_1(f)$ has been calculated, only the Midpoint Approximation $M_n = M_n(f)$ need be found. That is, we employ the following sequence of calculations:

$$T_1 = \frac{1}{2}(b-a)(f(a) + f(b));$$

$$\begin{array}{rll}
M_1 = (b-a)f(\frac{1}{2}(a+b)), & T_2 = \frac{1}{2}M_1 + \frac{1}{2}T_1, & S_2 = \frac{2}{3}M_1 + \frac{1}{3}T_1; \\
M_2, & T_4 = \frac{1}{2}M_2 + \frac{1}{2}T_2, & S_4 = \frac{2}{3}M_2 + \frac{1}{3}T_2; \\
M_4, & T_8 = \frac{1}{2}M_4 + \frac{1}{2}T_4, & S_8 = \frac{2}{3}M_4 + \frac{1}{3}T_4; \\
\ldots, & \ldots, & \ldots
\end{array}$$

Chapter 8

Sequences of Functions

8.1 Pointwise and Uniform Convergence

Definition 8.1.1. Let (f_n) be a sequence of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$, let $A_0 \subseteq A$, and let $f : A_0 \to \mathbb{R}$. We say that the **sequence** (f_n) **converges on** A_0 **to** f if, for each, $x \in A_0$, the sequence $(f_n(x))$ converges to $f(x)$ in $\mathbb{R}$. In this case we call f the **limit on** A_0 **of the sequence** (f_n). When such a function f exists, we say that the sequence (f_n) **is convergent on** A_0, or that (f_n) **converges pointwise on** A_0.

Lemma 8.1.1. A sequence (f_n) of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ converges to a function $f : A_0 \to \mathbb{R}$ on A_0 if and only if for each $\varepsilon > 0$ and each $x \in A_0$ there is a natural number $K(\varepsilon, x)$ such that if $n \geq K(\varepsilon, x)$, then

$$|f_n(x) - f(x)| < \varepsilon$$

Definition 8.1.2. A sequence (f_n) of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ **converges uniformly on** $A_0 \subseteq A$ to a function $f : A_0 \to \mathbb{R}$ if for each $\varepsilon > 0$ there is a natural number $K(\varepsilon)$ (depending on ε but **not** on $x \in A_0$) such that if $n \geq K(\varepsilon)$, then

$$|f_n(x) - f(x)| < \varepsilon \ \forall \ x \in A_0$$

In this case we say that the sequence (f_n) is **uniformly convergent on** A_0. Sometimes we write

$$f_n \rightrightarrows f \text{ on } A_0 \text{ or } f_n(x) \rightrightarrows f(x) \text{ for } x \in A_0$$

Lemma 8.1.2. A sequence (f_n) of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ does not converge uniformly on $A_0 \subseteq A$ to a function $f : A_0 \to \mathbb{R}$ if and only if for some $\varepsilon_0 > 0$ there is a subsequence (f_{n_k}) of (f_n) and a sequence (x_k) in A_0 such that

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0 \ \forall \ k \in \mathbb{N}$$

Definition 8.1.3. If $A \subseteq \mathbb{R}$ and $\varphi : A \to \mathbb{R}$ is a function, we say that φ is **bounded on** A if the set $\varphi(A)$ is a bounded subset of $\mathbb{R}$. If φ is bounded we define the **uniform norm of** φ **on** A by

$$||\varphi||_A := \sup\{|\varphi(x)| : x \in A\}$$

Note that it follows that if $\varepsilon > 0$, then

$$||\varphi||_A \leq \varepsilon \iff |\varphi(x)| \leq \varepsilon \ \forall \ x \in A$$

Lemma 8.1.3. A sequence (f_n) of bounded functions on $A \subseteq \mathbb{R}$ converges uniformly on A to f if and only if $||f_n - f||_A \to 0$.

Theorem 8.1.1 (Cauchy Criterion for Uniform Convergence). Let (f_n) be a sequence of bounded functions on $A \subseteq \mathbb{R}$. Then this sequence converges uniformly on A to a bounded function f if and only if for each $\varepsilon > 0$ there is a number $H(\varepsilon) \in \mathbb{N}$ such that for all $m, n \geq H(\varepsilon)$, then $||f_m - f_n||_A \leq \varepsilon$.

8.2 Interchange of Limits

Theorem 8.2.1. Let (f_n) be a sequence of continuous functions on a set $A \subseteq \mathbb{R}$ and suppose that (f_n) converges uniformly on A to a function $f : A \to \mathbb{R}$. Then f is continuous on A.

Remark 8.2.1. Although the uniform convergence of the sequence of continuous functions is sufficient to guarantee the continuity of the limit function, it is *not* necessary.

Theorem 8.2.2. Let $J \subseteq \mathbb{R}$ be a bounded interval and let (f_n) be a sequence of functions on J to $\mathbb{R}$. Suppose that there exists $x_0 \in J$ such that $(f_n(x_0))$ converges, and that the sequence (f_n') of derivatives exists on J and converges uniformly on J to a function g.
Then the sequence (f_n) converges uniformly on J to a function f that has a derivative at every point of J and $f' = g$.

Theorem 8.2.3. Let (f_n) be a sequence of functions in $\mathcal{R}[a, b]$ and suppose that (f_n) converges **uniformly** on $[a, b]$ to f. Then $f \in \mathcal{R}[a, b]$ and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$$

holds.

Theorem 8.2.4 (Bounded Convergence Theorem). Let (f_n) be a sequence in $\mathcal{R}[a, b]$ that converges on $[a, b]$ to a function $f \in \mathcal{R}[a, b]$.. Suppose also that there exists $B > 0$ such that $|f_n(x)| \leq B$ for all $x \in [a, b]$, $n \in \mathbb{N}$. Then

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$$

holds.

Theorem 8.2.5 (Dini's Theorem). Suppose that (f_n) is a monotone sequence of continuous functions on $I := [a, b]$ that converges on I to a continuous function f. Then the convergence of the sequence is uniform.

8.3 The Exponential and Logarithmic Functions

Theorem 8.3.1. There exists a function : $\mathbb{R} \to \mathbb{R}$ such that:

1. $E'(x) = E(x) \ \forall \ x \in \mathbb{R}$.

2. $E(0) = 1$.

Corollary 8.3.1. The function E has a derivative of every order and $E^{(n)}(x) = E(x)$ for all $n \in \mathbb{N}$, $x \in \mathbb{R}$.

Corollary 8.3.2. If $x > 0$, then $1 + x < E(x)$.

Theorem 8.3.2. The function $E : \mathbb{R} \to \mathbb{R}$ that satisfies (1) and (2) of *Theorem 8.3.1* is unique.

Theorem 8.3.3. The unique function $E : \mathbb{R} \to \mathbb{R}$, such that $E'(x) = E(x)$ for all $x \in \mathbb{R}$ and $E(0) = 1$, is called the **exponential function**. The number $e = E(1)$ is called **Euler's number**. We will frequently write

$$\exp(x) := E(x) \text{ or } e^x := E(x) \text{ for } x \in \mathbb{R}$$

Theorem 8.3.4. The exponential function satisfies the following properties:

1. $E(x) \neq 0$ for all $x \in \mathbb{R}$;

2. $E(x_+y) = E(x)E(y)$ for all $x, y, \in \mathbb{R}$.

3. $E(r) = e^r$ for all $r \in \mathbb{Q}$.

Theorem 8.3.5. The exponential function E is strictly increasing on $\mathbb{R}$ and has range equal to $\{y \in \mathbb{R} : y > 0\}$. Further, we have

$$\lim_{x \to -\infty} E(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} = \infty$$

Definition 8.3.1. The function inverse to $E : \mathbb{R} \to \mathbb{R}$ is called the **logarithm** (or the **natural logarithm**). It will be denoted by L, or by ln.

Theorem 8.3.6. The logarithm is a strictly increasing function L with domain $\{x \in \mathbb{R} : x > 0\}$ and range $\mathbb{R}$. The derivative of L is given by

1. $L'(x) = 1/x$ for $x > 0$.The logarithm satisfies the functional equation

2. $L(xy) = L(x) + L(y)$ for $x > 0, y > 0$. Moreover, we have

3. $L(1) = 0$ and $L(e) = 1$,

4. $L(x^r) = rL(x)$ for $x > 0$, $r \in \mathbb{Q}$,

5. $\lim_{x \to 0^+} L(x) = -\infty$ and $\lim_{x \to \infty} L(x) = \infty$

Definition 8.3.2. If $\alpha \in \mathbb{R}$ and $x > 0$, the number x^α is defined to be

$$x^\alpha := e^{\alpha \ln x} = E(\alpha L(x))$$

The function $x \mapsto x^\alpha$ for $x > 0$ is called the **power function** with exponent α.

Theorem 8.3.7. If $\alpha \in \mathbb{R}$ and x, y belong to $(0, \infty)$, then:

1. $1^\alpha = 1$

2. $x^\alpha > 0$

3. $(xy)^\alpha = x^\alpha y^\alpha$

4. $(x/y)^\alpha = x^\alpha / y^\alpha$.

Theorem 8.3.8. If $\alpha, \beta \in \mathbb{R}$ and $x \in (0, \infty)$, then:

1. $x^{\alpha+\beta} = x^\alpha x^\beta$

2. $(x^\alpha)^\beta = x^{\alpha\beta} = (x^\beta)^\alpha$

3. $x^{-\alpha} = 1/x^\alpha$

4. if $\alpha < \beta$, then $x^\alpha < x^\beta$ for $x > 1$.

Theorem 8.3.9. Let $\alpha \in \mathbb{R}$. Then the function $x \mapsto x^\alpha$ on $(0, \infty)$ to $\mathbb{R}$ is continuous and differentiable and

$$Dx^\alpha = \alpha x^{\alpha-1} \text{ for } x \in (0, \infty)$$

Definition 8.3.3. Let $a > 0,\ a \neq 1$. We define

$$\log_a(x) := \frac{\ln(x)}{\ln(a)} \text{ for } x \in (0, \infty)$$

For $x \in (0, \infty)$, the number $\log_a(x)$ is called the **logarithm of x to the base a**. The case $a = e$ yields the logarithm (or natural logarithm) function of *Definition 8.3.1*. The case $a = 10$ give sthe base 10 logarithm (or common logarithm) function $\log_{10}$ often used in computations. Properties of the functions $\log_a$ will be given in the exercises.

8.4 The Trigonometric Functions

Theorem 8.4.1. There exist functions $C : \mathbb{R} \to \mathbb{R}$ and $S : \mathbb{R} \to \mathbb{R}$ such that

1. $C''(x) = -C(x)$ and $S''(x) = -S(x)$ for all $x \in \mathbb{R}$.

2. $C(0) = 1,\ C'(0) = 0$, and $S(0) = 0,\ S'(0) = 1$.

Corollary 8.4.1. If $C,\ S$ are the functions in *Theorem 8.4.1*, then $C'(x) = -S(x)$ and $S'(x) = C(x)$ for all $x \in \mathbb{R}$. Moreover, these functions have derivatives of all orders.

Corollary 8.4.2. The functions C and S satisfy the Pythagorean Identity:

$$(C(x))^2 + (S(x))^2 = 1 \text{ for } x \in \mathbb{R}$$

Theorem 8.4.2. The functions C and S satisfying properties (1) and (2) of *Theorem 8.4.1* are unique.

Definition 8.4.1. The unique functions $C : \mathbb{R} \to \mathbb{R}$ and $S : \mathbb{R} \to \mathbb{R}$ such that $C''(x) = C(x)$ and $S''(x) = -S(x)$ for all $x \in \mathbb{R}$ and $C(0) = 1,\ C'(0) = 0$ and $S(0) = 0,\ S'(0) = 1$ are called the **cosine function** and the **sine function**, respectively. We ordinarily write

$$\cos x := C(x) \ \text{ and } \ \sin x := \S(x) \ \text{ for } \ x \in \mathbb{R}$$

Theorem 8.4.3. If $f : \mathbb{R} \to \mathbb{R}$ is such that

$$f''(x) = -f(x) \ \text{ for } \ x \in \mathbb{R}$$

then there exist real numbers $\alpha,\ \beta$ such that

$$f(x) = \alpha C(x) + \beta S(x) \ \text{ for } \ x \in \mathbb{R}$$

Theorem 8.4.4. The function C is even and S is odd in the sense that

1. $C(-x) = C(x)$ and $S(-x) = -S(x)$ for $x \in \mathbb{R}$. If $x,\ y \in \mathbb{R}$, then we have the "addition formulas".
2. $C(x + y) = C(x)C(y) - S(x)S(y)$, $S(x + y) = S(x)C(y) + C(x)S(y)$

Theorem 8.4.5. If $x \in \mathbb{R},\ x \geq 0$, then we have

1. $-x \leq S(x) \leq x$;
2. $1 - \frac{1}{2}x^2 \leq C(x) \leq 1$;
3. $x - \frac{1}{6}x^3 \leq S(x) \leq x$;
4. $1 - \frac{1}{2}x^2 \leq C(x) \leq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$.

Lemma 8.4.1. There exists a root γ of the cosine function in the interval $(\sqrt{2}, \sqrt{3})$. Moreover $C(x) > 0$ for $x \in [0, \gamma)$. The number 2γ is the smallest positive root of S.

Definition 8.4.2. Let $\pi := 2\gamma$ denote the smallest positive root of S.

Theorem 8.4.6. The functions C and S have period 2π in the sense that

1. $C(x + 2\pi) = C(x)$ and $S(x + 2\pi) = S(x)$ for $x \in \mathbb{R}$.
   Moreover we have
2. $S(x) = C(\frac{1}{2}\pi - x) = -C(x + \frac{1}{2}\pi)$, $C(x) = S(\frac{1}{2}\pi - x) = S(x + \frac{1}{2}\pi)$ for all $x \in \mathbb{R}$.

Chapter 9

Infinite Series

9.1 Absolute Convergence

Definition 9.1.1. Let $X := (x_n)$ be a sequence in $\mathbb{R}$. We say that the series $\sum x_n$ is **absolutely convergent** if the series $\sum |x_n|$ is convergent in $\mathbb{R}$. A series is said to be **conditionally** (or **nonabsolutely**) **convergent** if it is convergent, but it is not absolutely convergent.

Theorem 9.1.1. If a series in $\mathbb{R}$ is absolutely convergent, then it is convergent.

Theorem 9.1.2. If a series $\sum x_n$ is convergent, then any series obtained from it by grouping the terms is also convergent and to the same value.

Definition 9.1.2. A series $\sum y_k$ in $\mathbb{R}$ is a **rearrangement** of a series $\sum x_n$ if there is a bijection f of $\mathbb{N}$ onto $\mathbb{N}$ such that $y_k = x_{f(k)}$ for all $k \in \mathbb{N}$.

Theorem 9.1.3 (Rearrangement Theorem). Let $\sum x_n$ be an absolutely convergent series in $\mathbb{R}$. Then any rearrangement $\sum y_k$ of $\sum x_n$ converges to the same value.

Theorem 9.1.4. If $\sum a_n$ is conditionally convergent, then there exists a rearrangement of $\sum a_n$ such that

1. The rearrangement converges to any real number α
2. The rearrangement diverges to $\pm\infty$
3. The rearrangement oscillates between any two real numbers.

9.2 Tests for Absolute Convergence

Theorem 9.2.1 (Limit Comparison Test, II). Suppose that $X := (x_n)$ and $Y := (y_n)$ are nonzero real sequences and suppose that the following limit exists in $\mathbb{R}$:

$$r := \lim \left| \frac{x_n}{y_n} \right|$$

1. If $r \neq 0$, then $\sum x_n$ is absolutely convergent if and only if $\sum y_n$ is absolutely convergent.

2. If $r = 0$ and if $\sum y_n$ is absolutely convergent, then $\sum x_n$ is absolutely convergent.

Theorem 9.2.2 (Root Test). Let $X := (x_n)$ be a sequence in $\mathbb{R}$.

1. If there exist $r \in \mathbb{R}$ with $r < 1$ and $K \in \mathbb{N}$ such that

$$|x_n|^{1/n} \leq r \quad \text{for} \quad n \geq K,$$

then the series $\sum x_n$ is absolutely convergent.

2. If there exists $K \in \mathbb{N}$ such that

$$|x_n|^{1/n} \geq 1 \quad \text{for} \quad n \geq K,$$

then the series $\sum x_n$ is divergent.

Corollary 9.2.1. Let $X := (x_n)$ be a sequence in $\mathbb{R}$ and suppose that the limit

$$r := \lim |x_n|^{1/n}$$

exists in $\mathbb{R}$. Then $\sum x_n$ is absolutely convergent when $r < 1$ and is divergent when $r > 1$.

Theorem 9.2.3 (Ratio Test). Let $X := (x_n)$ be a sequence of nonzero real numbers.

1. If there exist $r \in \mathbb{R}$ with $0 < r < 1$ and $K \in \mathbb{N}$ such that

$$\left| \frac{x_{n+1}}{x_n} \right| \leq r \quad \text{for} \quad n \geq K,$$

then the series $\sum x_n$ is absolutely convergent.

2. If there exists $K \in \mathbb{N}$ such that

$$\left| \frac{x_{n+1}}{x_n} \right| \geq 1 \quad \text{for} \quad n \geq K,$$

then the series $\sum x_n$ is divergent.

Corollary 9.2.2. Let $X := (x_n)$ be a nonzero sequence in $\mathbb{R}$ and suppose that the limit

$$r := \lim \left| \frac{x_{n+1}}{x_n} \right|$$

exists in $\mathbb{R}$. Then $\sum x_n$ is absolutely convergent when $r < 1$ and is divergent when $r > 1$.

Theorem 9.2.4 (Integral Test). Let f be a positive, decreasing function on $\{t : t \geq 1\}$. Then the series $\sum_{k=1}^{\infty} f(k)$ converges if and only if the improper integral

$$\int_1^{\infty} f(t)dt = \lim_{b \to \infty} \int_1^b f(t)dt$$

exists. In the case of convergence, the partial sum $s_n = \sum_{k=1}^{n} f(k)$ and the sum $s = \sum_{k=1}^{\infty} f(k)$ satisfy the estimate

$$\int_{n+1}^{\infty} f(t)dt \leq s - s_n \leq \int_n^{\infty} f(t)dt$$

Theorem 9.2.5 (Raabe's Test). Let $X := (x_n)$ be a sequence of nonzero real numbers.

1. If there exist numbers $a > 1$ and $K \in \mathbb{N}$ such that

$$\left|\frac{x_{n+1}}{x_n}\right| \leq 1 - \frac{a}{n} \quad \text{for} \quad n \geq K,$$

   then $\sum x_n$ is absolutely convergent.

2. If there exist real numbers $a \leq 1$ and $K \in \mathbb{N}$ such that

$$\left|\frac{x_{n+1}}{x_n}\right| \geq 1 - \frac{a}{n} \quad \text{for} \quad n \geq K,$$

   then $\sum x_n$ is not absolutely convergent.

Corollary 9.2.3. Let $X := (x_n)$ be a nonzero sequence in $\mathbb{R}$ and let

$$a := \lim \left(n \left(1 - \left| \frac{x_{n+1}}{x_n} \right| \right) \right)$$

whenever this limit exists. Then $\sum x_n$ is absolutely convergent when $a > 1$ and is not absolutely convergent when $a < 1$.

9.3 Tests for Nonabsolute Convergence

Definition 9.3.1. A sequence $X := (x_n)$ of nonzero real numbers is said to be **alternating** if the terms $(-1)^{n+1}$, $n \in \mathbb{N}$, are all positive (or all negative) real numbers. If the sequence $X := (x_n)$ is alternative, we say that the series $\sum x_n$ it generates is an **alternating series**.

Theorem 9.3.1 (Alternating Series Test). Let $Z := (z_n)$ be a decreasing sequence of strictly positive numbers with $\lim(z_n) = 0$. Then the alternating series $\sum (-1)^{n+1} z_n$ is convergent.

Lemma 9.3.1 (Abel's Lemma). Let $X := (x_n)$ and $Y := (y_n)$ be sequences in $\mathbb{R}$ and let the partial sums of $\sum y_n$ be denoted by (s_n) with $s_0 := 0$. If $m > n$, then

$$\sum_{k=n+1}^{m} x_k y_k = (x_m s_m - x_{n+1} s_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) s_k$$

Theorem 9.3.2 (Dirichlet's Test). If $X := (x_n)$ is a decreasing sequence with $\lim x_n = 0$, and if the partial sums (s_n) of $\sum y_n$ are bounded, then the series $\sum x_n y_n$ is convergent.

Theorem 9.3.3 (Abel's Test). If $X := (x_n)$ is a convergent monotone sequence and the series $\sum y_n$ is convergent, then the series $\sum x_n y_n$ is also convergent.

9.4 Series of Functions

Definition 9.4.1. If (f_n) is a sequence of functions defined on a subset D of $\mathbb{R}$ with values in $\mathbb{R}$, the sequence of **partial sums** (s_n) of the infinite series $\sum f_n$ is defined for x in D by

$$\begin{aligned}
s_1(x) &:= f_1(x) \\
s_2(x) &:= s_1(x) + f_2(x) \\
&\cdots\cdots\cdots\cdots \\
s_{n+1}(x) &:= s_n(x) + f_{n+1}(x) \\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

In case the sequence (s_n) of functions converges on D to a function f, we say that the infinite series of functions $\sum f_n$ **converges** to f on D. We will often write

$$\sum f_n \quad \text{or} \quad \sum_{n=1}^{\infty} f_n$$

to denote either the series or the limit function, when it exists.

If the series $\sum |f_n(x)|$ converges for each x in D, we say that $\sum f_n$ is **absolutely convergent** on D. If the sequence (s_n) of partial sums is uniformly convergent on D to f, we say that $\sum f_n$ is **uniformly convergent** on D, or that it **converges to** f **uniformly on** D.

Theorem 9.4.1. If f_n is continuous on $D \subseteq \mathbb{R}$ to $\mathbb{R}$ for each $n \in \mathbb{N}$ and if $\sum f_n$ converges to f uniformly on D, then f is continuous on D.

Theorem 9.4.2. Suppose that the real-valued functions $f_n,\ n \in \mathbb{N}$ are Riemann integrable on the interval $J := [a, b]$. If the series $\sum f_n$ converges to f uniformly on J, then f is Riemann integrable and

$$\int_a^b f = \sum_{n=1}^{\infty} \int_a^b f_n$$

Theorem 9.4.3. For each $n \in \mathbb{N}$, let f_n be a real-valued function on $J := [a, b]$ that has a derivative f_n' on J. Suppose that the series $\sum f_n$ converges for at least one point of J and that the series of derivatives $\sum f_n'$ converges uniformly on J.
Then there exists a real-valued function f on J such that $\sum f_n$ converges uniformly on J to f. In addition, f has a derivative on J and $f' = \sum f_n'$.

Theorem 9.4.4 (Cauchy Criterion). Let (f_n) be a sequence of functions on $D \subseteq \mathbb{R}$ to $\mathbb{R}$. The series $\sum f_n$ is uniformly convergent on D if and only if for every $\varepsilon > 0$ there exists an $M(\varepsilon)$ such that if $m > n \geq M(\varepsilon)$, then

$$|f_{n+1}(x) + \cdots + f_m(x)| < \varepsilon \ \forall\ x \in D$$

Theorem 9.4.5 (Weierstrass M-Test). Let (M_n) be a sequence of positive real numbers such that $|f_n(x)| \leq M_n$ for $x \in D$, $n \in \mathbb{N}$. If the series $\sum M_n$ is convergent, then $\sum f_n$ is uniformly convergent on D, $\sum |f_n|$ is uniformly convergent on D, and $\sum f_n$ is absolutely convergent on D.

Definition 9.4.2. A series of real functions $\sum f_n$ is said to be a **power series around** $x = c$ if the function f_n has the form

$$f_n(x) = a_n(x - c)^n$$

where a_n and c belong to $\mathbb{R}$ and where $n = 0, 1, 2, \ldots$.

Definition 9.4.3. Let $\sum a_n x^n$ be a power series. If the sequence $(|a_n|^{1/n})$ is bounded, we set $\rho := \limsup(|a_n|^{1/n})$; if this sequence is not bounded we set $\rho = +\infty$. We define the **radius of convergence** of $\sum a_n x^n$ to be given by

$$R := \begin{cases} 0 & \text{if } \rho = +\infty \\ 1/\rho & \text{if } 0 < \rho < +\infty \\ +\infty & \text{if } \rho = 0 \end{cases}$$

The **interval of convergence** is the open interval $(-R, R)$.

Theorem 9.4.6 (Cauchy-Hadamard Theorem). If R is the radius of convergence of the power series $\sum a_n x^n$, then the series is absolutely convergent if $|x| < R$ and is convergent if $|x| > R$.

Remark 9.4.1. It will be noted that the Cauchy-Hadamard Theorem makes no statement as to whether the power series converges when $|x| = R$. Indeed, anything can happen, as the examples

$$\sum x^n, \ \sum \frac{1}{n} x^n, \ \sum \frac{1}{n^2} x^n$$

show. Since $\lim(n^{1/n}) = 1$, each of these power series has a radius of convergence equal to 1. The first power series converges at neither of the points $x = -1$ and $x = +1$; the second series converges at $x = -1$ but diverges at $x = +1$; and the third power series converges at both $x = -1$ and $x = +1$. (Find a power series with $R = 1$ that converges at $x = +1$ but diverges at $x = -1$.)

Theorem 9.4.7. Let R be the radius of convergence of $\sum a_n x^n$ and let K be a closed and bounded interval contained in the interval of convergence $(-R, R)$. Then the power series converges uniformly on K.

Theorem 9.4.8. Let $\sum a_n(x - c)^n$ be a power series. Then either

1. The series is absolutely convergent on $\mathbb{R}$
2. The series converges only at one point, $x = c$
3. There exists $R \in \mathbb{R}$ such that $\sum a_n(x - c)^n$ is absolutely convergent for all $|x - c| < R$, and is divergent for all $|x - c| > R$. (Note that the endpoints must be tested separately.)

Theorem 9.4.9. The limit of a power series is continuous on the interval of convergence. A power series can be integrated term-by-term over any closed and bounded interval contained in the interval of convergence.

Theorem 9.4.10 (Differentiation Theorem). A power series can be differentiated term-by-term within the interval of convergence. In fact, if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad \text{then} \quad f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{for} \quad |x| < R.$$

Both series have the same radius of convergence.

Remark 9.4.2. It is to be observed that the theorem makes no assertion about the endpoints of the interval of convergence. If a series is convergent at an endpoint, then the differentiated series may or may not be convergent at this point.

Theorem 9.4.11 (Uniqueness Theorem). If $\sum a_n x^n$ and $\sum b_n x^n$ converge on some interval $(-r, r)$, $r > 0$, to the same function f, then

$$a_n = b_n \ \forall \ n \in \mathbb{N}$$

The Taylor Series is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$$

Chapter 10

The Generalized Riemann Integral

10.1 Definition and Main Properties

In *Definition 5.2.2*, we defined a **gauge** on $[a,b]$ to be a strictly positive function $\delta : [a,b] \to (0,\infty)$. Further, a tagged partition $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^{n}$ of $[a,b]$, where $I_i := [x_{i-1}, x_i]$, is said to be **δ-fine** in the case

$$t_i \in I_i \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)] \quad \text{for} \quad i = 1, \ldots, n$$

Definition 10.1.1. A function $f : [a,b] \to \mathbb{R}$ is said to be ***generalized*** **Riemann integrable** on $[a,b]$ if there exists a number $L \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a gauge δ_ε on $[a,b]$ such that if $\dot{\mathcal{P}}$ is any δ_ε-fine partition of $[a,b]$, then

$$|S(f;\dot{\mathcal{P}}) - L| < \varepsilon$$

The collection of all generalized Riemann integrable functions will usually be denoted by $\mathcal{R}^*[a,b]$.

It will be shown that if $f \in \mathcal{R}^*[a,b]$, then the number L is uniquely determined; it will be called the **generalized Riemann integral of** f over $[a,b]$. It will also be shown that if $f \in \mathcal{R}[a,b]$, then $f \in \mathcal{R}^*[a,b]$ and the value of the two integrals is the same. Therefore, it will not cause any ambiguity if we also denote the generalized Riemann integral of $f \in \mathcal{R}^*[a,b]$ by the symbols

$$\int_a^b f \quad \text{or} \quad \int_a^b f(x)dx$$

Theorem 10.1.1 (Uniqueness Theorem). If $f \in \mathcal{R}^*[a,b]$, then the value of the integral is uniquely determined.

Theorem 10.1.2 (Consistency Theorem). If $f \in \mathcal{R}[a,b]$ with integral L, then also $f \in \mathcal{R}^*[a,b]$ with integral L.

Theorem 10.1.3. Suppose that f and g are in $\mathcal{R}^*[a,b]$. Then:

1. If $k \in \mathbb{R}$, the function kf is in $\mathcal{R}^*[a,b]$ and

$$\int_a^b kf = k\int_a^b f$$

2. The function $f+g$ is in $\mathcal{R}^*[a,b]$ and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

3. If $f(x) \le g(x)$ for all $x \in [a,b]$, then

$$\int_a^b f \le \int_a^b g$$

Theorem 10.1.4 (**Cauchy Criterion**)**.** A function $f : [a,b] \to \mathbb{R}$ belongs to $\mathcal{R}^*[a,b]$ if and only if for every $\varepsilon > 0$ there exists a gauge η_ε on $[a,b]$ such that if $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ are any partitions of $[a,b]$ that are η_ε-fine, then

$$|S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})| < \varepsilon$$

Theorem 10.1.5 (**Squeeze Theorem**)**.** Let $f : [a,b] \to \mathbb{R}$. Then $f \in \mathcal{R}^*[a,b]$ if and only if for every $\varepsilon > 0$ there exist functions α_ε and ω_ε in $\mathcal{R}^*[a,b]$ with

$$\alpha_\varepsilon(x) \le f(x) \le \omega_\varepsilon(x) \ \forall\ x \in [a,b]$$

and such that

$$\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) \le \varepsilon$$

Theorem 10.1.6 (**Additivity Theorem**)**.** Let $f : [a,b] \to \mathbb{R}$ and let $c \in (a,b)$. Then $f \in \mathcal{R}^*[a,b]$ if and only if its restrictions to $[a,c]$ and $[c,b]$ are both generalized Riemann integrable. In this case

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Theorem 10.1.7 (**The Fundamental Theorem of Calculus (First Form)**)**.** Suppose there exists a **countable** set E in $[a,b]$, and functions $f, F : [a,b] \to \mathbb{R}$ such that:

1. F is continuous on $[a,b]$.

2. $F'(x) = f(x)$ for all $x \in [a,b] \setminus E$.
Then f belongs to $\mathcal{R}^*[a,b]$ and

$$\int_a^b f = F(b) - F(a)$$

Theorem 10.1.8 (**Fundamental Theorem of Calculus (Second Form)**)**.** Let f belong to $\mathcal{R}^*[a,b]$ and let F be the indefinite integral of f. Then we have:

1. F is continuous on $[a,b]$.

2. There exists a null set Z such that if $x \in [a,b] \setminus Z$, then F is differentiable at x and $F'(x) = f(x)$.

3. If f is continuous at $c \in [a,b]$, then $F'(c) = f(c)$.

Theorem 10.1.9 (**Substitution Theorem**).

1. Let $I := [a,b]$ and $J := [\alpha,\beta]$, and let $F : I \to \mathbb{R}$ and $\varphi : J \to \mathbb{R}$ be continuous functions with $\varphi(J) \subseteq I$.

2. Suppose there exist sets $E_f \subset I$ and $E_\varphi \subset J$ such that $f(x) = F'(x)$ for $x \in I \setminus E_f$, that $\varphi'(t)$ exists for $t \in J \setminus E_\varphi$, and that $E := \varphi^{-1}(E_f) \cup E_\varphi$ is countable.

3. Set $f(x) := 0$ for $x \in E_f$ and $\varphi'(t) := 0$ for $t \in E_\varphi$. We conclude that $f \in \mathcal{R}^*(\varphi(J))$, that $(f \circ \varphi) \cdot \varphi^t \in \mathcal{R}^*(J)$ and that

$$\int_\alpha^\beta (f \circ \varphi) \cdot \varphi^t = F \circ \varphi \,|_\alpha^\beta = \int_{\varphi(\alpha)}^{\varphi(\beta)} f$$

Theorem 10.1.10 (**Multiplication Theorem**). If $f \in \mathcal{R}^*[a,b]$ and if g is a monotone function on $[a,b]$, then the product $f \cdot g$ belongs to $\mathcal{R}^*[a,b]$.

Theorem 10.1.11 (**Integration by Parts Theorem**). Let F and G be differentiable on $[a,b]$. Then $F'G$ belongs to $\mathcal{R}^*[a,b]$ if and only if FG' belongs to $\mathcal{R}^*[a,b]$. In this case we have

$$\int_a^b F'G = FG\,|_a^b - \int_a^b FG'$$

Theorem 10.1.12 (**Taylor's Theorem**). Suppose that $f, f', f'', \ldots, f^{(n)}$ and $f^{(n+1)}$ exist on $[a,b]$. Then we have

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + R_n$$

where the remainder is given by

$$R_n = \frac{1}{n!}\int_a^b f^{(n+1)}(t) \cdot (b-t)^n dt$$

10.2 Improper and Lebesgue Integrals

Theorem 10.2.1 (**Hake's Theorem**). If $f : [a,b] \to \mathbb{R}$, then $f \in \mathcal{R}^*[a,b]$ if and only if for every $\gamma \in (a,b)$ the restriction of f to $[a,\gamma]$ belongs to $\mathcal{R}^*[a,\gamma]$ and

$$\lim_{\gamma \to b^-} \int_a^\gamma f = A \in \mathbb{R}$$

In this case $\int_a^b f = A$.

Definition 10.2.1. A function $f \in \mathcal{R}^*[a,b]$ such that $|f| \in \mathcal{R}^*[a,b]$ is said to be **Lebesgue integrable** on $[a,b]$. The collection of all Lebesgue integrable functions on $[a,b]$ is denoted by $\mathcal{L}[a,b]$.

Theorem 10.2.2 (**Comparison Test**). If $f, \omega \in \mathcal{R}^*[a,b]$ and $|f(x)| \leq \omega(x)$ for all $x \in [a,b]$, then $f \in \mathcal{L}[a,b]$ and

$$\left| \int_a^b f \right| \leq \int_a^b |f| \leq \int_a^b \omega$$

Theorem 10.2.3. If $f, g \in \mathcal{L}[a,b]$ and if $c \in \mathbb{R}$, then cf and $f+g$ also belong to $\mathcal{L}[a,b]$. Moreover

$$\int_a^b cf = c\int_a^b f \quad \text{and} \quad \int_a^b |f+g| \leq \int_a^b |f| + \int_a^b |g|$$

Theorem 10.2.4. If $f \in \mathcal{R}^*[a,b]$, the following assertions are equivalent:

1. $f \in \mathcal{L}[a,b]$.
2. There exists $\omega \in \mathcal{L}[a,b]$ such that $f(x) \leq \omega(x)$ for all $x \in [a,b]$.
3. There exists $\alpha \in \mathcal{L}[a,b]$ such that $\alpha(x) \leq f(x)$ for all $x \in [a,b]$.

Theorem 10.2.5. If $f, g \in \mathcal{L}[a,b]$, then the functions $\max\{f,g\}$ and $\min\{f,g\}$ also belong to $\mathcal{L}[a,b]$.

Theorem 10.2.6. Suppose that f, g, α, and ω belong to $\mathcal{R}^*[a,b]$. If

$$f \leq \omega,\ g \leq \omega \quad \text{or if} \quad \alpha \leq f,\ \alpha \leq g,$$

then $\max\{f,g\}$ and $\min\{f,g\}$ also belong to $\mathcal{R}^*[a,b]$.

Definition 10.2.2. If $f \in \mathcal{L}[a,b]$, we define the **seminorm** of f to be

$$||f|| := \int_a^b |f|$$

If $f, g \in \mathcal{L}[a,b]$, we define the **distance between** f **and** g to be

$$\text{dist}(f,g) := ||f-g|| = \int_a^b |f-g|$$

Theorem 10.2.7. The seminorm function satisfies:

1. $||f|| \geq 0$ for all $f \in \mathcal{L}[a,b]$.
2. If $f(x) = 0$ for $x \in [a,b]$, then $||f|| = 0$.
3. If $f \in \mathcal{L}[a,b]$ and $c \in \mathbb{R}$, then $||cf|| = |c| \cdot ||f||$.
4. If $f, g \in \mathcal{L}[a,b]$, then $||f+g|| \leq ||f|| + ||g||$.

Theorem 10.2.8. The distance function satisfies:

1. $\text{dist}(f,g) \geq 0$ for all $f, g \in \mathcal{L}[a,b]$.
2. If $f(x) = g(x)$ for $x \in [a,b]$, then $\text{dist}(f,g) = 0$.

3. $\text{dist}(f, g) = \text{dist}(g, f)$ for all $f, g \in \mathcal{L}[a, b]$.

4. $\text{dist}(f, h) \le \text{dist}(f, g) + \text{dist}(g, h)$ for all $f, g, h \in \mathcal{L}[a, b]$.

Theorem 10.2.9 (**Completeness Theorem**)**.** A sequence (f_n) of functions in $\mathcal{L}[a, b]$ converges to a function $f \in \mathcal{L}[a, b]$ if and only if it has the property that for every $\varepsilon > 0$ there exists $H(\varepsilon)$ such that if $m, n \ge H(\varepsilon)$, then

$$||f_m - f_n|| = \text{dist}(f_m, f_n) < \varepsilon$$

10.3 Infinite Intervals

Definition 10.3.1.

1. A function $f : [a, \infty) \to \mathbb{R}$ is said to be **generalized Riemann integrable** if there exists $A \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a gauge δ_ε on $[a, \infty]$ such that if $\dot{\mathcal{P}}$ is any δ_ε-fine tagged subpartition of $[a, \infty)$, then $|S(f; \dot{\mathcal{P}}) - A| \le \varepsilon$. In this case, we write $f \in \mathcal{R}^*[a, \infty)$ and

$$\int_a^b f := A$$

2. A function $f : [a, \infty) \to \mathbb{R}$ is said to be **Lebesgue integrable** if both f and $|f|$ belong to $\mathcal{R}^*[a, \infty)$. In this case we write $f \in \mathcal{L}[a, \infty)$.

Theorem 10.3.1 (**Hake's Theorem**)**.** If $f : [a, \infty) \to \mathbb{R}$, then $f \in \mathcal{R}^*[a, \infty)$ if and only if for every $\gamma \in (a, \infty)$ the restriction of f to $[a, \gamma]$ belongs to $\mathcal{R}^*[a, \gamma]$ and

$$\lim_{\gamma \to \infty} \int_a^\gamma f = A \in \mathbb{R}$$

In this case $\displaystyle\int_a^\infty f = A$.

Theorem 10.3.2 (**Fundamental Theorem**)**.** Suppose that E is a countable subset of $[a, \infty)$ and that $f, F : [a, \infty) \to \mathbb{R}$ are such that:

1. F is continuous on $[a, \infty)$ and $\lim\limits_{x \to \infty} F(x)$ exists.

2. $F'(x) = f(x)$ for all $x \in (a, \infty),\ x \notin E$.
Then f belongs to $\mathcal{R}^*[a, \infty)$ and

$$\int_a^\infty f = \lim_{x \to \infty} F(x) - F(a).$$

Theorem 10.3.3 (**Hake's Theorem**)**.** If $h : (-\infty, \infty) \to \mathbb{R}$, then $h \in \mathcal{R}^*(-\infty, \infty)$ if and only if for every $\beta < \gamma$ in $(-\infty, \infty)$, the restriction of h to $[\beta, \gamma]$ is in $\mathcal{R}^*[\beta, \gamma]$ and

$$\lim_{\substack{\beta \to -\infty \\ \gamma \to +\infty}} \int_\beta^\gamma h = C \in \mathbb{R}$$

In this case $\displaystyle\int_{-\infty}^\infty h = C$.

Theorem 10.3.4 (Fundamental Theorem). Suppose that E is a countable subset of $(-\infty, \infty)$ and that $h, H : (-\infty, \infty) \to \mathbb{R}$ satisfy:

1. H is continuous on $(-\infty, \infty)$ and the limits $\lim_{x \to \pm\infty} H(x)$ exist.

2. $H'(x) = h(x)$ for all $x \in (-\infty, \infty),\ x \notin E$.

Then h belongs to $\mathcal{R}^*(-\infty, \infty)$ and

$$\int_{-\infty}^{\infty} h = \lim_{x \to \infty} H(x) - \lim_{\gamma \to -\infty} H(\gamma)$$

10.4 Convergence Theorems

Theorem 10.4.1 (Uniform Convergence Theorem). Let (f_k) be a sequence in $\mathcal{R}^*[a, b]$ and suppose that (f_k) converges **uniformly** on $[a, b]$ to f. Then $f \in \mathcal{R}^*[a, b]$ and

$$\int_a^b f = \lim_{k \to \infty} \int_a^b f_k$$

holds.

Definition 10.4.1. A sequence (f_k) in $\mathcal{R}^*(I)$ is said to be **equi-integrable** if for every $\varepsilon > 0$ there exists a gauge δ_ε on I such that if $\dot{\mathcal{P}}$ is any δ_ε-fine partition of I and $k \in \mathbb{N}$, then $\left| S(f_k; \dot{\mathcal{P}}) - \int_I f_k \right| < \varepsilon$.

Theorem 10.4.2 (Equi-integrability Theorem). If $(f_k) \in \mathcal{R}^*(I)$ is equi-integrable on I and if $f(x) = \lim f_k(x)$ for all $x \in I$, then $f \in \mathcal{R}^*(I)$ and

$$\int_I f = \lim_{k \to \infty} \int_I f_k$$

Definition 10.4.2. We say that a sequence of functions on an interval $I \subseteq \mathbb{R}$ is **monotone increasing** if it satisfies $f_1(x) \le f_2(x) \le \cdots \le f_k(x) \le f_{k+1}(x) \le \ldots$ for all $k \in \mathbb{N}$, $x \in I$. It is said to be **monotone decreasing** if it satisfies the opposite string of inequalities, and to be **monotone** if it is either monotone increasing or decreasing.

Theorem 10.4.3 (Monotone Convergence Theorem). Let (f_k) be a monotone sequence of functions in $\mathcal{R}^*(I)$ such that $f(x) = \lim f_k(x)$ almost everywhere on I. Then $f \in \mathcal{R}^*(I)$ if and only if the sequence of integrals $\left(\int_I f_k\right)$ is bounded in $\mathbb{R}$, in which case

$$\int_I f = \lim_{k \to \infty} \int_I f_k.$$

Theorem 10.4.4 (Dominated Convergence Theorem). Let (f_n) be a sequence in $\mathcal{R}^*(I)$ and let $f(x) = \lim f_k(x)$ almost everywhere on I. If there exist functions α, ω in $\mathcal{R}^*(I)$ such that

$$\alpha(x) \le f_k(x) \le \omega(x) \quad \text{for almost every} \quad x \in I$$

then $f \in \mathcal{R}^*(I)$ and

$$\int_I f = \lim_{k\to\infty} \int_I f_k.$$

Moreover, if α and ω belong to $\mathcal{L}(I)$, then f_k and f belong to $\mathcal{L}(I)$ and

$$||f_k - f|| = \int_I |f_k - f| \to 0$$

Definition 10.4.3. A function $f : [a, b] \to \mathbb{R}$ is said to be **(Lebesgue) measurable** if there exists a sequence (s_k) of step functions on $[a, b]$ such that

$$f(x) = \lim_{k\to\infty} s_k(x) \quad \text{for almost every} \quad x \in [a, b]$$

We denote the collection of measurable functions on $[a, b]$ by $\mathcal{M}[a, b]$.

Theorem 10.4.5. Let f and g belong to $\mathcal{M}[a, b]$ and let $c \in \mathbb{R}$.

1. Then the functions $cf, |f|, f + g, f - g$, and $f \cdot g$ also belong to $\mathcal{M}[a, b]$.
2. If $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous, then the composition $\varphi \circ f \in \mathcal{M}[a, b]$.
3. If (f_n) is a sequence in $\mathcal{M}[a, b]$ and $f(x) = \lim f_n(x)$ almost everywhere on I, then $f \in \mathcal{M}[a, b]$.

Theorem 10.4.6. A function $f : [a, b] \to \mathbb{R}$ is in $\mathcal{M}[a, b]$ if and only if there exists a sequence (g_k) of continuous functions such that

$$f(x) = \lim_{k\to\infty} g(x) \quad \text{for almost every} \quad x \in [a, b]$$

Theorem 10.4.7 (**Measurability Theorem**)**.** If $f \in \mathcal{R}^*[a, b]$, then $f \in \mathcal{M}[a, b]$

Theorem 10.4.8 (**Integrability Theorem**)**.** Let $f \in \mathcal{M}[a, b]$. Then $f \in \mathcal{R}^*[a, b]$ if and only if there exist functions $\alpha, \omega \in \mathcal{R}^*[a, b]$ such that

$$\alpha(x) \leq f(x) \leq \omega(x) \quad \text{for almost every} \quad x \in [a, b]$$

Moreover, if either α or ω belongs to $\mathcal{L}[a, b]$, then $f \in \mathcal{L}[a, b]$.

Chapter 11

A Glimpse Into Topology

11.1 Open and Closed Sets in $\mathbb{R}$

Definition 11.1.1. A **neighborhood** of a point $x \in \mathbb{R}$ is any set V that contains an ε-neighborhood $V_\varepsilon(x) := (x - \varepsilon, x + \varepsilon)$ of x for some $\varepsilon > 0$.

Definition 11.1.2.

1. A subset G of $\mathbb{R}$ is **open** in $\mathbb{R}$ if for each $x \in G$ there exists a neighborhood V of x such that $V \subseteq G$.

2. A subset F of $\mathbb{R}$ is **closed in** $\mathbb{R}$ if the complement $\mathcal{C}(F) := \mathbb{R} \setminus F$ is open in $\mathbb{R}$.

Theorem 11.1.1 (Open Set Properties).

1. The union of an arbitrary collection of open subsets in $\mathbb{R}$ is open.

2. The intersection of any finite collection of open sets in $\mathbb{R}$ is open.

Theorem 11.1.2 (Closed Set Properties).

1. The intersection of an arbitrary collection of closed sets in $\mathbb{R}$ is closed.

2. The union of any finite collection of closed sets in $\mathbb{R}$ is closed.

Theorem 11.1.3 (Characterization of Closed Sets). Let $F \subset \mathbb{R}$; then the following assertions are equivalent:

1. F is a closed subset of $\mathbb{R}$.

2. If $X = (x_n)$ is any convergent sequence of elements in F, then $\lim X$ belongs to F.

Theorem 11.1.4. A subset of $\mathbb{R}$ is closed if and only if it contains all of its cluster points.

Theorem 11.1.5. A subset of $\mathbb{R}$ is open if and only if it is the union of countably many disjoint open intervals in $\mathbb{R}$.

Definition 11.1.3. The **Cantor Set** $\mathbb{F}$ is the intersection of the sets $F_n, n \in \mathbb{N}$, obtained by successive removal of open middle thirds, starting with $[0, 1]$.

11.2 Compact Sets

Definition 11.2.1. Let A be a subset of $\mathbb{R}$. An **open cover** of A is a collection $\mathcal{G} = \{G_\alpha\}$ of open sets in $\mathbb{R}$ whose union contains A; that is,

$$A \subseteq \bigcup_\alpha G_\alpha$$

If $\mathcal{G}'$ is a subcollection of sets from $\mathcal{G}$ such that the union of the sets in $\mathcal{G}'$ also contains A, then $\mathcal{G}'$ is called a **subcover** of $\mathcal{G}$. If $\mathcal{G}'$ consists of finitely many sets, then we call $\mathcal{G}'$ a **finite subcover** of $\mathcal{G}$.

Definition 11.2.2. A subset K of $\mathbb{R}$ is said to be **compact** if *every* open cover of K has a finite subcover.

Theorem 11.2.1. If K is a compact subset of $\mathbb{R}$, then K is closed and bounded.

Theorem 11.2.2 (Heine-Borel Theorem). A subset K of $\mathbb{R}$ is compact if and only if it is closed and bounded.

Theorem 11.2.3. A subset K of $\mathbb{R}$ is compact if and only if every sequence in K has a subsequence that converges to a point in K.

11.3 Continuous Functions

Lemma 11.3.1. A function $f : A \to \mathbb{R}$ is continuous at the point c in A if and only if for every neighborhood U of $f(c)$, there exists a neighborhood V of c such that if $x \in V \cap A$, then $f(x) \in U$.

Theorem 11.3.1 (Global Continuity Theorem). Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a function with domain A. Then the following are equivalent:

1. f is continuous at every point of A.
2. For every open set G in $\mathbb{R}$, there exists an open set H in $\mathbb{R}$ such that $H \cap A = f^{-1}(G)$.

Corollary 11.3.1. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if $f^{-1}(G)$ is open in $\mathbb{R}$ whenever G is open.

Theorem 11.3.2 (Preservation of Compactness). If K is a compact subset of $\mathbb{R}$ and if $f : K \to \mathbb{R}$ is continuous on K, then $f(K)$ is compact.

Theorem 11.3.3. If K is a compact subset of $\mathbb{R}$ and $f : K \to \mathbb{R}$ is injective and continuous, then f^{-1} is continuous on $f(K)$.

Theorem 11.3.4. Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$. Then the following are equivalent:

1. f is continuous on A
2. *Theorem 5.1.3:* $\forall$ converging sequences $x_n \to a \in A$, then $f(x_n) \to f(a)$
3. *Theorem 11.3.2:* For each open set $U \subseteq \mathbb{R}$, $f^{-1}(U) \subseteq A$ is open relative to A. (i.e. inverse images of open sets are relatively open to A. $f^{-1}(U) = V \cap A$ for some open set $V \subseteq \mathbb{R}$)
4. For each closed set $F \subseteq \mathbb{R}$, then $f^{-1}(F) \subseteq A$ is closed relative to A. (i.e. inverse images of closed sets are relatively closed)

11.4 Metric Spaces

Definition 11.4.1. A **metric** on a set S is a function $d : S \times S \to \mathbb{R}$ that satisfies the following properties:

1. $d(x, y) \geq 0 \; \forall \; x, y \in S$ *(positivity)*;

2. $d(x, y) = 0$ if and only if $x = y$ *(definiteness)*;

3. $d(x, y) = d(y, x) \; \forall \; x, y \in S$ *(symmetry)*;

4. $d(x, y) \leq d(x, z) + d(z, y) \; \forall \; x, y, z \in S$ *(triangle inequality)*

A **metric space** (S, d) is a set S together with a metric d on S.

Definition 11.4.2. Let (S, d) be a metric space. Then for $\varepsilon > 0$, the **ε-neighborhood** of a point x_0 in S is the set

$$V_\varepsilon(x_0) := \{x \in S : d(x_0, x) < \varepsilon\}$$

A **neighborhood** of x_0 is any set U that contains an ε-neighborhood of x_0 for some $\varepsilon > 0$.

Definition 11.4.3. Let (x_n) be a sequence in the metric space (S, d). The sequence (x_n) is said to **converge** to x in S if for any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that $x_n \in V_\varepsilon(x)$ for all $n \geq K$.

Definition 11.4.4. Let (S, d) be a metric space. A sequence (x_n) in S is said to be a **Cauchy sequence** if for each $\varepsilon > 0$, there exists $H \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq H$.

Definition 11.4.5. A metric space (S, d) is said to be **complete** if each Cauchy sequence in S converges to a point of S.

Definition 11.4.6. Let (S, d) be a metric space. A subset G of S is said to be an **open** set in S if for every point $x \in S$ there is a neighborhood U of x such that $U \subseteq G$. A subset F of S is said to be a **closed** set in S if the complement $S \setminus F$ is an open set in S.

Definition 11.4.7. Let (S_1, d_1) and (S_2, d_2) be metric spaces, and let $f : S_1 \to S_2$ be a function from S_1 to S_2. The function f is said to be **continuous** at the point c in S_1 if for every ε-neighborhood $V_\varepsilon(f(c))$ of $f(c)$ there exists a δ-neighborhood $V_\delta(c)$ of c such that if $x \in V_\delta(c)$, then $f(x) \in V_\varepsilon(f(c))$.

Theorem 11.4.1 (Global Continuity Theorem). If (S_1, d_1) and (S_2, d_2) are metric spaces, then a function $f : S_1 \to S_2$ is continuous on S_1 if and only if $f^{-1}(G)$ is open in S_1 whenever G is open in S_2.

Theorem 11.4.2 (Preservation of Compactness). If (S, d) is a compact metric space and if the function $f : S \to \mathbb{R}$ is continuous, then $f(S)$ is compact in $\mathbb{R}$.

Definition 11.4.8. A **semimetric** on a set S is a function $d : S \times S \to \mathbb{R}$ that satisfies all of the conditions of a metric, except that condition (2) is replaces by the weaker condition

$$d(x, y) = 0 \quad \text{if} \quad x = y$$

A **semimetric space** (S, d) is a set S together with a semimetric d on S.