Linear Algebra Theorems and Definitions

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January 9, 2024

All theorems, corollaries, lemmas, remarks, and asides are direct quotes from
Linear Algebra, 4th Edition, by Stephen H. Friedberg, Arnold J. Insel, and Lawrence E. Spence

Contents

Chapter 0

List of Symbols

A_{ij}	the ij-th entry of the matrix A
A^{-1}	the inverse of the matrix A
$A^{\dagger}$	the pseudoinverse of the matrix A
A^*	the adjoint of the matrix A
$\tilde{A}_{ij}$	the matrix A with row i and column j deleted
A^t	the transpose of the matrix A
$(A\|B)$	the matrix A augmented by the matrix B
$B_1 \bigoplus \cdots \bigoplus B_k$	the direct sum of matrices B_1 through B_k
$\mathcal{B}(V)$	the set of bilinear forms on V
β^*	the dual basis of β
β_x	the T-cyclic basis generated by x
$\mathbb{C}$	the field of complex numbers
$\mathbb{C}_i$	the ith Gerschgorin disk
$\operatorname{cond}(A)$	the condition number of the matrix A
$C^n(\mathbb{R})$	set of functions f on $\mathbb{R}$ with $f^{(n)}$ continuous
C^∞	set of functions with derivatives of every order
$C(\mathbb{R})$	the vector space of continuous functions on $\mathbb{R}$
$C([0,1])$	the vector space of continuous functions on $[0,1]$
C_x	the T-cyclic subspaces generated by x
D	the derivative operator on C^∞
$\det(A)$	the determinant of the matrix A
δ_{ij}	the Kronecker delta
$\dim(V)$	the dimension of V
e^A	$\lim_{m\to\infty}\left(I + A + \frac{A^2}{2!} + \cdots + \frac{A^m}{m!}\right)$
e_i	the ith standard vector of $\mathbb{F}^n$

E_λ	the eigenspace of T corresponding to λ
$\mathbb{F}$	a field
$f(A)$	the polynomial $f(x)$ evaluated at the matrix A
F^n	the set of n-tuples with entries in a field $\mathbb{F}$
$f(T)$	the polynomial $f(x)$ evaluated at the operator T
$\mathcal{F}(S, \mathbb{F})$	the set of functions from S to a field $\mathbb{F}$
H	space of continuous complex functions on $[0, 2\pi]$
I_n or I	the $n \times n$ identity matrix
$\mathbb{I}_V$ or $\mathbb{I}$	the identity operator on V
K_λ	generalized eigenspace of T corresponding to λ
K_ϕ	$\{x \mid (\phi(T))^p(x) = 0,$ for some positive integer $p\}$
L_A	left-multiplication transformation by matrix A
$\lim\limits_{m\to\infty} A_m$	the limit of a sequence of matrices
$\mathcal{L}(V)$	the space of linear transformations from V to V
$\mathcal{L}(V, W)$	the space of linear transformations from V to W
$M_{m\times n}(\mathbb{F})$	the set of $m \times n$ matrices with entries in $\mathbb{F}$
$v(A)$	the column sum of the matrix A
$v_j(A)$	the jth column sum of the matrix A
$N(T)$	the null space of T
nullity (T)	the dimension of the null space of T
O	the zero matrix
per (M)	the permanent of the 2×2 matrix M
$P(\mathbb{F})$	the space of polynomials with coefficients in $\mathbb{F}$
$P_n(\mathbb{F})$	the polynomials in $P(\mathbb{F})$ of degree at most n
ϕ_β	the standard representation with respect to basis β
$\mathbb{R}$	the field of real numbers
rank (A)	the rank of the matrix A
rank (T)	the rank of the linear transformation T
$\rho(A)$	the row sum of the matrix A
$\rho_i(A)$	the ith row sum of the matrix A
$R(T)$	the range of the linear transformation T
$S_1 + S_2$	the sum of sets S_1 and S_2
span (S)	the span of the set S
$S^\perp$	the orthogonal complement of the set S
$[T]_\beta$	the matrix representation of T in basis β
$[T]_\beta^\gamma$	the matrix representation of T in bases β and γ
T^{-1}	the inverse of the linear transformation T

$T^\dagger$	the pseudoinverse of the linear transformation T
T^*	the adjoint of the linear operator T
T_0	the zero transformation
T^t	the transpose of the linear transformation T
T_θ	the rotation transformation by θ
T_W	the restriction of T to a subspace W
$\mathrm{tr}\,(A)$	the trace of the matrix A
V^*	the dual space of the vector space V
V/W	the quotient space of V modulo W
$W_1 + \cdots + W_k$	the sum of subspaces W_1 through W_k
$\sum_{i=1}^{k} W_i$	the sum of subspaces W_i through W_k
$W_1 \bigoplus W_2$	the direct sum of subspaces W_1 and W_2
$W_1 \bigoplus \cdots \bigoplus W_k$	the direct sum of subspaces W_1 through W_k
$\|x\|$	the norm of the vector $\vec{x}$
$[x]_\beta$	the coordinate vector of x relative to β
$\langle x, y \rangle$	the inner product of $\vec{x}$ and $\vec{y}$
$\mathbb{Z}_2$	the field consisting of 0 and 1
$\overline{\vec{z}}$	the complex conjugate of $\vec{z}$
$\vec{0}$	the zero vector

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