Linear Algebra Theorems and Definitions

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All theorems, corollaries, lemmas, remarks, and asides are direct quotes from
Linear Algebra, 4th Edition, by Stephen H. Friedberg, Arnold J. Insel, and Lawrence E. Spence

Contents

Chapter 0

List of Symbols

A_{ij}	the ij-th entry of the matrix A
A^{-1}	the inverse of the matrix A
$A^{\dagger}$	the pseudoinverse of the matrix A
A^{*}	the adjoint of the matrix A
$\tilde{A}_{ij}$	the matrix A with row i and column j deleted
A^{t}	the transpose of the matrix A
$(A\|B)$	the matrix A augmented by the matrix B
$B_1 \bigoplus \cdots \bigoplus B_k$	the direct sum of matrices B_1 through B_k
$\mathcal{B}(V)$	the set of bilinear forms on V
β^{*}	the dual basis of β
β_x	the T-cyclic basis generated by x
$\mathbb{C}$	the field of complex numbers
$\mathbb{C}_i$	the ith Gerschgorin disk
$\mathrm{cond}\,(A)$	the condition number of the matrix A
$C^n(\mathbb{R})$	set of functions f on $\mathbb{R}$ with $f^{(n)}$ continuous
C^{∞}	set of functions with derivatives of every order
$C(\mathbb{R})$	the vector space of continuous functions on $\mathbb{R}$
$C([0,1])$	the vector space of continuous functions on $[0,1]$
C_x	the T-cyclic subspaces generated by x
D	the derivative operator on C^{∞}
$\det\,(A)$	the determinant of the matrix A
δ_{ij}	the Kronecker delta
$\dim\,(V)$	the dimension of V
e^{A}	$\lim_{m\to\infty}\left(I + A + \frac{A^2}{2!} + \cdots + \frac{A^m}{m!}\right)$
e_i	the ith standard vector of $\mathbb{F}^n$

E_λ	the eigenspace of T corresponding to λ
$\mathbb{F}$	a field
$f(A)$	the polynomial $f(x)$ evaluated at the matrix A
F^n	the set of n-tuples with entries in a field $\mathbb{F}$
$f(T)$	the polynomial $f(x)$ evaluated at the operator T
$\mathcal{F}(S, \mathbb{F})$	the set of functions from S to a field $\mathbb{F}$
H	space of continuous complex functions on $[0, 2\pi]$
I_n or I	the $n \times n$ identity matrix
$\mathbb{I}_V$ or $\mathbb{I}$	the identity operator on V
K_λ	generalized eigenspace of T corresponding to λ
K_ϕ	$\{x \mid (\phi(T))^p(x) = 0$, for some positive integer $p\}$
L_A	left-multiplication transformation by matrix A
$\lim\limits_{m\to\infty} A_m$	the limit of a sequence of matrices
$\mathcal{L}(V)$	the space of linear transformations from V to V
$\mathcal{L}(V, W)$	the space of linear transformations from V to W
$M_{m\times n}(\mathbb{F})$	the set of $m \times n$ matrices with entries in $\mathbb{F}$
$v(A)$	the column sum of the matrix A
$v_j(A)$	the jth column sum of the matrix A
$N(T)$	the null space of T
nullity (T)	the dimension of the null space of T
O	the zero matrix
per (M)	the permanent of the 2×2 matrix M
$P(\mathbb{F})$	the space of polynomials with coefficients in $\mathbb{F}$
$P_n(\mathbb{F})$	the polynomials in $P(\mathbb{F})$ of degree at most n
ϕ_β	the standard representation with respect to basis β
$\mathbb{R}$	the field of real numbers
rank (A)	the rank of the matrix A
rank (T)	the rank of the linear transformation T
$\rho(A)$	the row sum of the matrix A
$\rho_i(A)$	the ith row sum of the matrix A
$R(T)$	the range of the linear transformation T
$S_1 + S_2$	the sum of sets S_1 and S_2
span (S)	the span of the set S
$S^\perp$	the orthogonal complement of the set S
$[T]_\beta$	the matrix representation of T in basis β
$[T]_\beta^\gamma$	the matrix representation of T in bases β and γ
T^{-1}	the inverse of the linear transformation T

$T^\dagger$	the pseudoinverse of the linear transformation T
T^*	the adjoint of the linear operator T
T_0	the zero transformation
T^t	the transpose of the linear transformation T
T_θ	the rotation transformation by θ
T_W	the restriction of T to a subspace W
$\mathrm{tr}\,(A)$	the trace of the matrix A
V^*	the dual space of the vector space V
V/W	the quotient space of V modulo W
$W_1 + \cdots + W_k$	the sum of subspaces W_1 through W_k
$\sum_{i=1}^{k} W_i$	the sum of subspaces W_i through W_k
$W_1 \bigoplus W_2$	the direct sum of subspaces W_1 and W_2
$W_1 \bigoplus \cdots \bigoplus W_k$	the direct sum of subspaces W_1 through W_k
$\|x\|$	the norm of the vector $\vec{x}$
$[x]_\beta$	the coordinate vector of x relative to β
$\langle x, y \rangle$	the inner product of $\vec{x}$ and $\vec{y}$
$\mathbb{Z}_2$	the field consisting of 0 and 1
$\overline{\vec{z}}$	the complex conjugate of $\vec{z}$
$\vec{0}$	the zero vector

Chapter 1

Vector Spaces

1.1 Introduction

Theorem 1.1.1 (**Parallelogram Law for Vector Addition**).
The sum of two vectors x and y that act at the same point P is the vector beginning at P that is represented by the diagonal of parallelogram having x and y as adjacent sides.

Definition 1.1.1.
Two nonzero vectors x and y are called **parallel** if $y = tx$ for some nonzero real number t. (Thus nonzero vectors having the same or opposite directions are parallel.)

1.2 Vector Spaces

Definition 1.2.1.
A **vector space** (or **linear space**) V over a field $\mathbb{F}$ consists of a set on which two operations (called **addition** and **scalar multiplication**, respectively) are defined so that for each pair of elements x and y in V there is a unique element a in $\mathbb{F}$ and each element x in V there is a unique element ax in V, such that the following conditions hold:

(VS 1) For all x, y in V, $x + y = y + x$ (commutativity of addition).

(VS 2) For all x, y in V, $(x + y) + z = x + (y + z)$ (associativity of addition).

(VS 3) There exists an element in V denoted by 0 such that $x + 0 = x$ for each x in V.

(VS 4) For each element x in V there exists an element y in V such that $x + y = 0$.

(VS 5) For each element x in V, $1x = x$.

(VS 6) For each pair of elements a, b in $\mathbb{F}$ and each element x in V, $(ab)x = a(bx)$.

(VS 7) For each element a in $\mathbb{F}$ and each pair of elements x, y in V, $a(x + y) = ax + ay$.

(VS 8) For each pair of elements a, b in $\mathbb{F}$ and each element x in V, $(a + b)x = ax + bx$.

The elements $x + y$ and ax are called the **sum** of x and y and the **product** of a and x, respectively.

The elements of the field $\mathbb{F}$ are called **scalars** and the elements of the vector space V are called **vectors**.

Note: The reader should not confuse this use of the word "vector" with the physical entity discussed in section 1.1: the word "vector" is now being used to describe any element of a vector space.

Definition 1.2.2.
An object of the form $(a_1, a_2, \ldots, a_n)$, where the entries $a_1, a_2, \ldots, a_n$ are elements of a field $\mathbb{F}$, is called an ***n*-tuple** with entries from $\mathbb{F}$. The elements $a_1, a_2, \ldots, a_n$ are called the **entries** or **components** of the n-tuple. Two n-tuples $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ with entries from a field $\mathbb{F}$ are called **equal** if $a_i = b_i$ for $i = 1, 2, \ldots, n$.

Definition 1.2.3.
Vectors in $\mathbb{F}^n$ may be written as **column vectors**

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

rather than as **row vectors** $(a_1, a_2, \ldots, a_n)$. Since a 1-tuple whose only entry is from $\mathbb{F}$ can be regarded as an element of $\mathbb{F}$, we usually write $\mathbb{F}$ rather than $\mathbb{F}^1$ for the vector space of 1-tuples with entry from $\mathbb{F}$.

Definition 1.2.4.
An $m \times n$ **matrix** with entries from a field $\mathbb{F}$ is a rectangular array of the form

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix},$$

where each entry a_{ij} $(1 \leq i \leq m,\ 1 \leq j \leq n)$ is an element of $\mathbb{F}$. We call the entries a_{ij} with $i = j$ the **diagonal entries** of the matrix. The entries $a_{i1}, a_{i2}, \ldots, a_{in}$ compose the ***i*th row** of the matrix, and the entries $a_{1j}, a_{2j}, \ldots, a_{mj}$ compose the ***j*th column** of the matrix. The rows of the preceding matrix are regarded as vectors in $\mathbb{F}^n$, and the columns are regarded as vectors in $\mathbb{F}^m$. The $m \times n$ matrix in which each entry equals zero is called the **zero matrix** and is denoted by O.

In this book, we denote matrices by capital italic letters (e.g. A, B, and C), and we denote the entry of a matrix A that lies in row i and column j by A_{ij}. In addition, if the number of rows and columns of a matrix are equal, the matrix is called **square**.

Two $m \times n$ matrices A and B are called **equal** if all their corresponding entries are equal, that is, if $A_{ij} = B_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Definition 1.2.5.
The set of all $m \times n$ matrices with entries from a field $\mathbb{F}$ is a vector space which we denote by $M_{m\times n}(\mathbb{F})$, with the following operations of **matrix addition** and **scalar multiplication**: For $A, B \in M_{m\times n}(\mathbb{F})$ and $c \in \mathbb{F}$,

$$(A+B)_{ij} = A_{ij} + B_{ij} \quad \text{and} \quad (cA)_{ij} = cA_{ij}$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Definition 1.2.6.
Let S be any nonempty set and $\mathbb{F}$ be any field, and let $\mathcal{F}(S, \mathbb{F})$ denote the set of all functions from S to $\mathbb{F}$. Two functions f and g in $\mathcal{F}(S, \mathbb{F})$ are called **equal** if $f(s) = g(s)$ for each $s \in S$. The set $\mathcal{F}(S, \mathbb{F})$ is a vector space with the operations of addition and scalar multiplication defined for $f, g \in \mathcal{F}(S, \mathbb{F})$ and $c \in \mathbb{F}$ defined by

$$(f+g)(s) = f(s) + g(s) \quad \text{and} \quad (cf)(s) = c[f(s)]$$

for each $s \in S$. Note that these are the familiar operations of addition and scalar multiplication for functions used in algebra and calculus.

Definition 1.2.7.
A **polynomial** with coefficients from a field $\mathbb{F}$ is an expression of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a nonnegative integer and each a_k, called the **coefficient** of x^k, is in $\mathbb{F}$. If $f(x) = 0$, that is, if $a_n = a_{n-1} = \cdots = a_0 = 0$, then $f(x)$ is called the **zero polynomial** and, for convenience, its degree is defined to be -1; otherwise, the **degree** of a polynomial is defined to be the largest exponent of x that appears in the representation

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

with a nonzero coefficient. Note that the polynomials of degree zero may be written in the form $f(x) = c$ for some nonzero scalar c. Two polynomials,

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

and

$$g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0,$$

are called **equal** if $m = n$ and $a_i = b_i$ for $i = 1, 2, \ldots, n$.

Definition 1.2.8.
Let $\mathbb{F}$ be any field. A **sequence** in $\mathbb{F}$ is a function σ from the positive integers into $\mathbb{F}$. In this book, the sequence σ such that $\sigma(n) = a_n$ for $n = 1, 2, \ldots$ is denoted $\{a_n\}$. Let V consist of all sequences $\{a_n\}$ in $\mathbb{F}$ that have only a finite number of nonzero terms a_n. If $\{a_n\}$ and $\{b_n\}$ are in V and $t \in \mathbb{F}$, define

$$\{a_n\} + \{b_n\} = \{a_n + b_n\} \quad \text{and} \quad t\{a_n\} = \{ta_n\}$$

Theorem 1.2.1 (Cancellation Law for Vector Addition).
If x, y and z are vectors in a vector space V such that $x + z = y + z$, then $x = y$.

Corollary 1.2.1.
The vector 0 described in (VS 3) is unique.

Corollary 1.2.2.
The vector y described in (VS 4) is unique.

Definition 1.2.9.
The vector 0 in (VS 3) is called the **zero vector** of V, and the vector y in (VS 4) (that is, the unique vector such that $x + y = 0$) is called the **additive inverse** of x and is denoted by $-x$.

Theorem 1.2.2.
In any vector space V, the following statements are true:

1. $0x = 0$ for each $x \in V$.
2. $(-a)x = -(ax) = a(-x)$ for each $a \in \mathbb{F}$ and each $x \in V$.
3. $a0 = 0$ for each $a \in \mathbb{F}$.

Definition 1.2.10.
Let $V = \{0\}$ consist of a single vector 0 and define $0 + 0 = 0$ and $c0 = 0$ for each scalar $c \in \mathbb{F}$. Then V is called the **zero vector space**.

Definition 1.2.11.
A real-valued function f defined on the real line is called an **even function** if $f(-t) = f(t)$ for each real number t, and is called an **odd function** if $f(-t) = -f(t)$ for each real number t.

Definition 1.2.12.
If S_1 and S_2 are nonempty subsets of a vector space V, then the **sum** of S_1 and S_2, denoted $S_1 + S_2$, is the set $\{x + y \mid x \in S_1, \text{ and } y \in S_2\}$.

Definition 1.2.13.
A vector space V is called the **direct sum** of W_1 and W_2 if W_1 and W_2 are subspaces of V such that $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$. We denote that V is the direct sum of W_1 and W_2 by writing $V = W_1 \oplus W_2$.

Definition 1.2.14.
A matrix M is called **skew-symmetric** if $M^t = -M$.

Definition 1.2.15.
Let W be a subspace of a vector space V over a field $\mathbb{F}$. For any $v \in V$, the set $\{v\} + W = \{v + w \mid w \in W\}$ is called the **coset of** W **containing** v. It is customary to denote this coset by $v + W$ rather than $\{v\} + W$.

Definition 1.2.16.
Let W be a subspace of a vector space V over a field $\mathbb{F}$, and let $S := \{v + W \mid v \in V\}$ be the set of all cosets of W. Then S is called the **quotient space of V modulo W**, and is denoted by V/W. Addition and scalar multiplication by the scalars of $\mathbb{F}$ can be defined as follows:

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$$

for all $v_1, v_2 \in V$, and

$$a(v + W) = av + W$$

for all $v \in V$ and $a \in \mathbb{F}$.

1.3 Subspaces

Definition 1.3.1.
A subset W of a vector space V over a field $\mathbb{F}$ is called a **subspace** of V if W is a vector space over $\mathbb{F}$ with the operations of addition and scalar multiplication defined on V.

In any vector space V, note that V and $\{0\}$ are subspaces. The latter is called the **zero subspace** of V.

Fortunately, it is not necessary to verify all of the vector space properties to prove that a subset is a subspace. Because properties (VS 1), (VS 2), (VS 5), (VS 6), (VS 7) and (VS 8) hold for all vectors in the vector space, these properties automatically hold for the vectors in any subset. Thus a subset W of a vector space V is a subspace of V if and only if the following four properties hold:

1. $x + y \in W$ whenever $x \in W$ and $y \in W$. (W is **closed under addition**).

2. $cx \in W$ whenever $c \in \mathbb{F}$ and $x \in W$. (W is **closed under scalar multiplication**).

3. W has a zero vector.

4. Each vector in W has an additive inverse in W.1

Theorem 1.3.1.
Let V be a vector space and W a subset of V. Then W is a subspace of V if and only if the following three conditions hold for the operations defined in V.

1. $0 \in W$.

2. $x + y \in W$ whenever $x \in W$ and $y \in W$.

3. $cx \in W$ whenever $c \in \mathbb{F}$ and $x \in W$.

Definition 1.3.2.
The **transpose** A^t of an $m \times n$ matrix A is the $n \times m$ matrix obtained from A by interchanging the rows with the columns; that is, $(A^t)_{ij} = A_{ji}$.

Definition 1.3.3.
A **symmetric matrix** is a matrix A such that $A^t = A$.

Definition 1.3.4.
An $n \times n$ matrix M is called a **diagonal matrix** if $M_{ij} = 0$ whenever $i \neq j$; that is, if all nondiagonal entries are zero.

Definition 1.3.5.
The **trace** of an $n \times n$ matrix M, denoted $\mathrm{tr}(M)$, is the sum of the diagonal entries of M; that is,

$$\mathrm{tr}(M) = M_{11} + M_{22} + \cdots + M_{nn}.$$

Theorem 1.3.2.
Any intersection of subspaces of a vector space V is a subspace of V.

Definition 1.3.6.
An $m \times n$ matrix A is called **upper triangular** if all entries lying below the diagonal entries are zero; that is, if $A_{ij} = 0$ whenever $i > j$.

1.4 Linear Combinations and Systems of Linear Equations

Definition 1.4.1.
Let V be a vector space and S a nonempty subset of V. A vector $v \in V$ is called a **linear combination** of vectors of S if there exist a finite number of vectors $v_1, v_2, \ldots, v_n$ in S and scalars $a_1, a_2, \ldots, a_n$ in $\mathbb{F}$ such that $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$. In this case we also say that v is a linear combination of $v_1, v_2, \ldots, v_n$ and call $a_1, a_2, \ldots, a_n$ the **coefficients** of the linear combination.

Definition 1.4.2.
Let S be a nonempty subset of a vector space V. The **span** of S, denoted $\mathrm{span}(S)$, is the set consisting of all linear combinations of the vectors in S. For convenience, we define $\mathrm{span}(\emptyset) = \{0\}$.

Theorem 1.4.1.
The span of any subset S of a vector space V is a subspace of V. Moreover, any subspace of V that contains S must also contain the span of S.

Definition 1.4.3.
A subset S of a vector space V **generates** (or **spans**) V if $\mathrm{span}(S) = V$. In this case, we also say that the vectors of S generate (or span) V.

1.5 Linear Dependence and Linear Independence

Definition 1.5.1.
A subset S of a vector space V is called **linearly dependent** if there exist a finite number of distinct vectors $v_1, v_2, \ldots, v_n$ in S and scalars $a_1, a_2, \ldots, a_n$ not all zero, such that

$$a_1 v_2 + a_2 v_2 + \cdots + a_n v_n = 0$$

In this case, we also say that the vectors of S are linearly dependent.

For any vectors $v_1, v_2, \ldots, v_n$, we have $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$ if $a_1 = a_2 = \cdots = a_n = 0$. We call this the **trivial representation** of 0 as a linear combination of $v_1, v_2, \ldots, v_n$. Thus, for a set tot be linearly dependent, there must exist a nontrivial representation of 0 as a linear combination of vectors in the set. Consequently, any subset of a vector space that contains the zero vector is linearly dependent, because $0 = 1 \cdot 0$ is a nontrivial representation of 0 as a linear combination of vectors in the set.

Definition 1.5.2.
A subset S of a vector space that is not linearly dependent is called **linearly independent**. As before, we also say that the vectors of S are linearly independent.

The following facts about linearly independent sets are true in any vector space.

1. The empty set is linearly independent, for linearly dependent sets must be nonempty.

2. A set consisting of a single nonzero vector is linearly independent. For if $\{v\}$ is linearly dependent, then $av = 0$ for some nonzero scalar a. thus

$$v = a^{-1}(av) = a^{-1}0 = 0.$$

3. A set is linearly independent if and only if the only representations of 0 as linear combinations of its vectors are trivial representations.

Theorem 1.5.1.
Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_1 is linearly dependent, then S_2 is linearly dependent.

Corollary 1.5.1.
Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_2 is linearly independent, then S_1 is linearly independent.

Theorem 1.5.2.
Let S be a linearly independent subset of a vector space V, and let v be a vector in V that is not in S. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in \text{span}(S)$.

1.6 Bases and Dimension

Definition 1.6.1.
A **basis** β for a vector space V is a linearly independent subset of V that generates V. If β is a basis for V, we also say that the vectors of β form a basis for V.

Theorem 1.6.1.
Let V be a vector space and $\beta = \{v_1, v_2, \ldots, v_n\}$ be a subset of V. Then β is a basis for V if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors of β, that is, can be expressed in the form

$$v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$$

for unique scalars $a_1, a_2, \ldots, a_n$.

Theorem 1.6.2.
If a vector space V is generated by a finite set S, then some subset of S is a basis for V. Hence V has a finite basis.

Theorem 1.6.3 (Replacement Theorem).
Let V be a vector space that is generated by a set G containing exactly n vectors, and let L be a linearly independent subset of V containing exactly m vectors. Then $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ generates V.

Corollary 1.6.1. Let V be a vector space having a finite basis. Then every basis for V contains the same number of vectors.

Definition 1.6.2.
A vector space is called **finite-dimensional** if it has a basis consisting of a finite number of vectors. The unique number of vectors in each basis for V is called the **dimension** of V and is denoted by $\dim(V)$. A vector space that is not finite-dimensional is called **infinite-dimensional**.

Corollary 1.6.2.
Let V be a vector space with dimension n.

1. Any finite generating set for V contains at least n vectors, and a generating set for V that contains exactly n vectors is a basis for V.

2. Any linearly independent subset of V that contains exactly n vectors is a basis for V.

3. Every linearly independent subset of V can be extended to a basis for V.

Theorem 1.6.4.
Let W be a subspace of a finite-dimensional vector space V. Then W is finite-dimensional and $\dim(W) \leq \dim(V)$. Moreover, if $\dim(W) = \dim(V)$, then $V = W$.

Corollary 1.6.3.
If W is a subspace of a finite-dimensional vector space V, then any basis for W can be extended to a basis for V.

Definition 1.6.3 (**The Lagrange Interpolation Formula**).
Corollary 2 of the replacement theorem can be applied to obtain a useful formula. Let $c_0, c_1, \ldots, c_n$ be distinct scalars in an infinite field $\mathbb{F}$. The polynomials $f_0(x), f_1(x), \ldots, f_n(x)$ defined by

$$f_i(x) = \frac{(x - c_0)\ldots(x - c_{i-1})(x - c_{i+1})\ldots(x - c_n)}{(c_i - c_0)\ldots(c_i - c_{i-1})(c_i - c_{i+1})\ldots(c_i - c_n)} = \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{x - c_k}{c_i - c_k}$$

are called the **Lagrange polynomials** (associated with $c_0, c_1, \ldots, c_n$). Note that each $f_i(x)$ is a polynomial of degree n and hence is in $P_n(\mathbb{F})$. By regarding $f_i(x)$ as a polynomial function $f_i : \mathbb{F} \to \mathbb{F}$, we see that

$$f_i(c_j) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases} \tag{1.1}$$

This property of Lagrange polynomials can be used to show that $\beta = \{f_0, f_1, \ldots, f_n\}$ is a linearly independent subset of $P_n(\mathbb{F})$. Suppose that

$$\sum_{i=0}^{n} a_i f_i = 0 \text{ for some scalars } a_0, a_1, \ldots, a_n,$$

where 0 denotes the zero function. Then

$$\sum_{i=0}^{n} a_i f_i(c_j) = 0 \text{ for } j = 0, 1, \ldots, n.$$

But also

$$\sum_{i=0}^{n} a_i f_i(c_j) = a_j$$

by (1.1). Hence $a_j = 0$ for $j = 0, 1, \ldots, n$; so β is linearly independent. Since the dimension of $P_n(\mathbb{F})$ is $n + 1$, it follows from Corollary 2 of the replacement theorem that β is a basis for $P_n(\mathbb{F})$.

Because β is a basis for $P_n(\mathbb{F})$, every polynomial function g in $P_n(\mathbb{F})$ is a linear combination of polynomial functions of β, say,

$$g = \sum_{i=0}^{n} b_i f_i.$$

It follows that

$$g(c_j) = \sum_{i=0}^{n} b_i f_i(c_j) = b_j;$$

so

$$g = \sum_{i=0}^{n} g(c_i) f_i$$

is the unique representation of g as a linear combination of elements of β. This representation is called the **Lagrange interpolation formula**. Notice that the preceding argument shows that if $b_0, b_1, \ldots, b_n$ are any $n+1$ scalars in $\mathbb{F}$ (not necessarily distinct), then the polynomial function

$$g = \sum_{i=0}^{n} b_i f_i$$

is the unique polynomial in $P_n(\mathbb{F})$ such that $g(c_j) = b_j$. Thus we have found the unique polynomial of degree not exceeding n that has specified values b_j at given points c_j in its domain $(j = 0, 1, \ldots, n)$.

An important consequence of the Lagrange interpolation formula is the following result: If $f \in P_n(\mathbb{F})$ and $f(c_i) = 0$, for $n+1$ distinct scalars $c_0, c_1, \ldots, c_n$ in $\mathbb{F}$, then f is the zero function.

1.7 Maximal Linearly Independent Subsets

Definition 1.7.1.
Let $\mathcal{F}$ be a family of sets. A member M of $\mathcal{F}$ is called **maximal** (with respect to set inclusion) if M is contained in no member of $\mathcal{F}$ other than M itself.

Definition 1.7.2.
Let $\mathcal{F}$ be the family of all subsets of a nonempty set S. This family $\mathcal{F}$ is called the **power set** of S.

Definition 1.7.3.
A collection of sets $\mathcal{C}$ is called a **chain** (or **nest** or **tower**) if for each pair of sets A and B in $\mathcal{C}$, either $A \subseteq B$ or $B \subseteq A$.

Definition 1.7.4 (Maximal Principle).
Let $\mathcal{F}$ be a family of sets. If, for each chain $\mathcal{C} \subseteq \mathcal{F}$, there exists a member of $\mathcal{F}$ that contains each member of $\mathcal{C}$, then $\mathcal{F}$ contains a maximal member.

Note: The *Maximal Principle* is logically equivalent to the *Axiom of Choice*, which is an assumption in most axiomatic developments of set theory.

Definition 1.7.5.
Let S be a subset of a vector space V. A **maximal linearly independent subset** of S is a subset B of S satisfying both of the following conditions

1. B is linearly independent.

2. The only linearly independent subset of S that contains B is B itself.

Corollary 1.7.1.
Every vector space has a basis.

Chapter 2

Linear Transformations and Matrices

2.1 Linear Transformations, Null Spaces, and Ranges

2.2 The Matrix Representation of a Linear Transformation

2.3 Compositions of Linear Transformations and Matrix Multiplication

2.4 Invertibility and Isomorphisms

2.5 The Change of Coordinate Matrix

2.6 Dual Spaces

2.7 Homogeneous Linear Differential Equations with Constant Coefficients

Chapter 3

Elementary Matrix Operations and Systems of Linear Equations

3.1 Elementary Matrix Operations and Elementary Matrices

3.2 The Rank of a Matrix and Matrix Inverses

3.3 Systems of Linear Equations – Theoretical Aspects

3.4 Systems of Linear Equations – Computational Aspects

Chapter 4

Determinants

4.1 Determinants of Order 2

4.2 Determinants of Order n

4.3 Properties of Determinants

4.4 Summary – Important Facts about Determinants

4.5 A Characterization of the Determinant

Chapter 5

Diagonalization

5.1 Eigenvalues and Eigenvectors

5.2 Diagonalizability

5.3 Matrix Limits and Markov Chains

5.4 Invariant Subspaces and the Cayley-Hamilton Theorem

Chapter 6

Inner Product Spaces

6.1 Inner Products and Norms

6.2 The Gram-Schmidt Orthogonalization Process and Orthogonal Complements

6.3 The Adjoint of a Linear Operator

6.4 Normal and Self-Adjoing Operators

6.5 Unitary and Orthogonal Operators and Their Matrices

6.6 Orthogonal Projections and the Spectral Theorem

6.7 The Singular Value Decomposition and the Pseudoinverse

6.8 Bilinear and Quadratic Forms

6.9 Einstein's Special Theory of Relativity

6.10 Conditioning and the Rayleigh Quotient

6.11 The Geometry of Orthogonal Operators

Chapter 7

Canonical Forms

7.1 The Jordan Canonical Form I

7.2 The Jordan Canonical Form II

7.3 The Minimal Polynomial

7.4 The rational Canonical Form