Real Analysis Homework 4

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1. Section 3.1

   5) Use the definition of the limit of a sequence to establish the following limits.

      (a) $\lim(\frac{n}{n^2+1}) = 0$

      Recall the definition of the limit of a sequence:

      $$\text{A sequence converges to a limit } A \text{ if } \forall\ \varepsilon > 0,\ \exists N_\varepsilon \text{ s.t. } |a_n - A| < \varepsilon\ \forall\ n \geq N_\varepsilon,$$

      $$n \in \mathbb{N},\ N_\varepsilon \in \mathbb{N},\ \text{written } \lim_{n\to\infty} a_n = A$$

      Thus, what we ultimately want to show here is that $|\frac{n}{n^2+1} - 0| < \varepsilon, \forall\ n \geq N_\varepsilon$.

      Let's first take care of the denominator. We want to maximize the size of the denominator. So, we have

      $$n^2 + 1 > n^2\ \forall\ n \in \mathbb{N},\ \text{thus we have that } \frac{1}{n^2+1} < \frac{1}{n^2}\ \forall\ n \in \mathbb{N}$$

      Since there's no way to maximize the size of the numerator from what it currently is, combining both the numerator and denominator, we have

      $$\frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n}\ \forall\ n \in \mathbb{N}$$

      Now, given that $\varepsilon > 0$, we know that by Corollary 2.4.5 (If $t > 0$, then $\exists n_t \in \mathbb{N}$ s.t. $0 < \frac{1}{n_t} < t$), we know that $\exists N_\varepsilon \in \mathbb{N}$ s.t. $0 < \frac{1}{N_\varepsilon} < \varepsilon$, where $n = \varepsilon$. Let N_0 be the smallest of these numbers with this property. Then if $n \geq N_0$, $\frac{1}{n} < \frac{1}{N_0} < \varepsilon$. Thus we have

      $$\left|\frac{n}{n^2+1} - 0\right| = \left|\frac{n}{n^2+1}\right| = \frac{n}{n^2+1} < \frac{1}{n} < \varepsilon$$

      $\therefore \lim_{n\to\infty}(\frac{n}{n^2+1}) = 0$.

      (c) $\lim(\frac{3n+1}{2n+5}) = \frac{3}{2}$

      We want to show that $\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| < \varepsilon,\ \forall\ n \geq N_\varepsilon,\ n \in \mathbb{N},\ N_\varepsilon \in \mathbb{N},\ \forall\ \varepsilon > 0$

Since $n \in \mathbb{N}$, we know that $\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| = \frac{3n+1}{2n+5} - \frac{3}{2}$. So,

$$\frac{3n+1}{2n+5} - \frac{3}{2} = \frac{6n+2}{4n+10} - \frac{6n+15}{4n+10} = -\frac{13}{4n+10} < 0$$

and

$$\frac{13}{4n+10} < \frac{13}{4n} < \frac{16}{4n} < \frac{4}{n} \; \forall \; n \in \mathbb{N}$$

Given $\varepsilon > 0$, and since $\frac{4}{n} < \varepsilon \implies \frac{n}{4} > \varepsilon$, we know that by Corollary 2.4.5 ($t > 0 \implies \exists n_t \in \mathbb{N}$ s.t. $0 < \frac{1}{n_t} < t$), then $\exists N_\varepsilon \in \mathbb{N}$ s.t. $0 < \frac{1}{N_\varepsilon} < \frac{\varepsilon}{4}$, where $n = \varepsilon$. Let $N_0 \in \mathbb{N}$ be the smallest of these numbers with this property. Thus, if $n \geq N_0$, we have that $\frac{1}{n} < \frac{1}{N_0} < \frac{\varepsilon}{4}$. This gives us that $\frac{1}{n} < \frac{\varepsilon}{4} = \frac{4}{n} < \varepsilon$. So

$$\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| = \left|-\frac{13}{4n+10}\right| = \frac{13}{4n+10} < \frac{4}{n} < \varepsilon$$

$\therefore \lim(\frac{3n+1}{2n+5}) = \frac{3}{2}$.

(d) $\lim(\frac{n^2-1}{2n^2+3}) = \frac{1}{2}$

We want to show that $\left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| < \varepsilon, \; \forall \; \varepsilon > 0, \; \forall \; n \geq N_\varepsilon$, where $n, N_\varepsilon \in \mathbb{N}$. Since

$$\frac{n^2-1}{2n^2+3} - \frac{1}{2} = \frac{2n^2-2}{4n^2+6} - \frac{2n^2+3}{4n^2+6} = \frac{-5}{4n^2+6} < 0$$

So,

$$\left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| = \left|\frac{2n^2-2}{4n^2+6} - \frac{2n^2+3}{4n^2+6}\right| = \left|\frac{-5}{4n^2+6}\right| = \frac{5}{4n^2+6} < \frac{5}{4n^2} < \frac{5}{4n} < \frac{5}{n}$$

$\forall n \in \mathbb{N}$

Given $\varepsilon > 0$, and since $\frac{5}{n}, \varepsilon \implies \frac{n}{5} > \varepsilon$, we know that by Corollary 2.4.5 ($t > 0 \implies \exists n_t \in \mathbb{N}$ s.t. $0 < \frac{1}{n_t} < t$), then $\exists N_\varepsilon \in \mathbb{N}$ s.t. $0 < \frac{1}{N_\varepsilon} < \frac{\varepsilon}{5}$, where $n = \varepsilon$. Let N_0 be the smallest of these numbers with this property. Then if $n \geq N_0$, we have that $\frac{1}{n} < \frac{1}{n_0} < \frac{\varepsilon}{5}$. This yields $\frac{1}{n} < \frac{\varepsilon}{5} = \frac{5}{n} < \varepsilon$. So

$$\left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| = \left|-\frac{5}{4n^2+6}\right| = \frac{5}{4n^2+6} < \frac{5}{n} < \varepsilon$$

$\therefore \lim(\frac{n^2-1}{2n^2+3}) = \frac{1}{2}$

6) Show that

(a) $\lim(\frac{1}{\sqrt{n+7}}) = 0$

We want to show that $\left|\frac{1}{\sqrt{n+7}} - 0\right| < \varepsilon, \; \forall \; \varepsilon > 0, \; \forall \; n \geq N_\varepsilon$, where $n, N_\varepsilon \in \mathbb{N}$. Since $n + 7 > n \; \forall \; n \in \mathbb{N}$, we have that $\sqrt{n+7} > \sqrt{n} \; \forall \; n \in \mathbb{N}$ and therefore,

$\frac{1}{\sqrt{n+7}} < \frac{1}{\sqrt{n}}\ \forall\ n \in \mathbb{N}$.

Given $\varepsilon > 0$, we know by Corollary 2.4.5 ($t > 0 \implies \exists n_t$ s.t. $0 < \frac{1}{n_t} < t$), then $\exists N_\varepsilon \in \mathbb{N}$ s.t. $0 < \frac{1}{N_\varepsilon} < \varepsilon^2$. Thus we have that $\frac{1}{\sqrt{N_\varepsilon}} < \varepsilon$.

Thus if $n \geq N_\varepsilon$, we have that $\sqrt{N_\varepsilon} \leq \sqrt{n}$ which gives us that $\frac{1}{\sqrt{n+7}} < \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N_\varepsilon}} < \varepsilon$. Therefore we have $\left|\frac{1}{\sqrt{n+7}} - 0\right| = \left|\frac{1}{n+7}\right| = \frac{1}{\sqrt{n+7}} < \varepsilon\ \forall\ n \geq N_\varepsilon$.

$\therefore\ \lim(\frac{1}{\sqrt{n+7}}) = 0$

9) Show that if $x_n \geq 0\ \forall\ n \in \mathbb{N}$ and $\lim(x_n)$, then $\lim(\sqrt{x_n}) = 0$.

Let (x_n) be a sequence such that $\forall\ n \in \mathbb{N},\ x_n \geq 0$ and $\lim(x_n)$. We want to show that $\lim(\sqrt{x_n}) = 0$.

Let $\varepsilon > 0$. By the definition of the limit of a sequence, we know that $\exists N_\varepsilon \in \mathbb{N}$ s.t. $\forall\ n \geq N_\varepsilon$, the following inequality holds:

$$|x_n - 0| = |x_n| = x_n \geq 0 = x_n < \varepsilon^2$$

Thus, if $n \geq N_\varepsilon$, we have that

$$|\sqrt{x_n} - 0| = |\sqrt{x_n}| = \sqrt{x_n} > \sqrt{\varepsilon^2} = \varepsilon$$

and by the definition of the limit of a sequence, again, we have that $\lim(\sqrt{x_n}) = 0$.

11) Show that $\lim(\frac{1}{n} - \frac{1}{n+1}) = 0$.

We want to show the following:

$$\left|\left(\frac{1}{n} - \frac{1}{n+1}\right) - 0\right| < \varepsilon,\ \forall\ \varepsilon > 0,\ \forall\ n \geq N_\varepsilon,\ \text{for } n, N_\varepsilon \in \mathbb{N}$$

Recall Theorem 3.1.10:

Theorem 3.1.10. *Let (x_n) be a sequence of real numbers and let $x \in \mathbb{R}$. If (a_n) is a sequence of positive real numbers with $\lim(a_n) = 0$ and if for some constant $C > 0$ and some $m \in \mathbb{N}$ we have*

$$|x_n - x| \leq C a_n\ \forall\ n \geq m,$$

then it follows that $\lim(x_n) = x$.

We want to find a constant $C > 0$ and a sequence (a_n) such that $a_n > 0,\ \lim(a_n) = 0$, and

$$\left|\left(\frac{1}{n} - \frac{1}{n+1}\right) - 0\right| \leq C a_n\ \forall\ n \geq m,\ \text{for some } m \in \mathbb{N}$$

Let's first find C and a_n.

$$\begin{aligned}
\left|\left(\frac{1}{n} - \frac{1}{n+1}\right) - 0\right| &= \left|\frac{1}{n} - \frac{1}{n+1}\right| \\
&= \frac{1}{n} - \frac{1}{n+1} \qquad\qquad \left(\frac{1}{n} > \frac{1}{n+1}\right) \\
&= \frac{(n+1)-n}{n(n+1)} \\
&= \frac{1}{n^2+n}
\end{aligned}$$

Since $n < n + n^2 \ \forall\ n$, we have that $\frac{1}{n^2+n} < \frac{1}{n}$. Thus, choose $C = 1$, $(a_n) = \frac{1}{n}$ and $m = 1$.

Furthermore, we proved in class that $\lim(\frac{1}{n}) = 0$.

Now that all of the conditions of Theorem 3.1.10 have been satisfied, apply it to our original sequence of $x_n = \frac{1}{n} - \frac{1}{n+1}$, which yields:

$$\lim\left(\frac{1}{n} - \frac{1}{n+1}\right) = 0$$

12) Show that $\lim(\sqrt{n^2+1} - n) = 0$.

Let $\varepsilon > 0$ be given. Then, we have the following:

$$\begin{aligned}
\left|(\sqrt{n^2+1} - n) - 0\right| &= \left|\sqrt{n^2+1} - n\right| \\
&= \left|(\sqrt{n^2+1} - n) * \frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}\right| \\
&= \left|\frac{1}{\sqrt{n^2+1}+n}\right|
\end{aligned}$$

Now, let's look at the denominator, $\sqrt{n^2+1}+n$. Then, we know that $n^2+1 > n^2$. So, we have

$$\begin{aligned}
n^2+1 &> n^2 \\
\sqrt{n^2+1} &> n \\
\sqrt{n^2+1}+n &\geq 2n \\
\frac{1}{\sqrt{n^2+1}+n} &\leq \frac{1}{2n} = \varepsilon
\end{aligned}$$

Choose $N \geq \frac{1}{2\varepsilon}$.

Thus we have that

$$\forall\ \varepsilon > 0, \text{ choose } N \geq \frac{1}{2\varepsilon} \text{ then } |a_n - A| < \varepsilon\ \forall\ n \geq N$$

13) Show that $\lim(\frac{1}{3^n}) = 0$.

Since $n \leq 3n \iff \frac{1}{3^n} \leq \frac{1}{n}$, we have

$$\left|\frac{1}{3^n} - 0\right| \leq \frac{1}{n}$$

Using Theorem 3.1.10 and $\lim \frac{1}{n} = 0$ we get

$$\lim \frac{1}{3^n} = 0$$

2. Section 3.2

1) For x_n given by the following formulas, establish either the convergence or the divergence of the sequence $X = (x_n)$.

(a) $x_n := \frac{n}{n+1}$

$$x_n = \frac{1}{n+1} = \frac{1}{1+\frac{1}{n}}$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+\lim_{n\to\infty}\frac{1}{n}} = 1$$

therefore we have that the sequence $\{x_n\}$ converges to 1.

2) Give an example of two divergent sequences X and Y such that:

(a) their sum $X + Y$ converges,

Let $X = (0, 1, 0, 1, 0, 1, ...)$ and let $Y = (1, 0, 1, 0, 1, 0, ...)$. Clearly X and Y are divergent because the difference of two consecutive terms is equal to 1. Thus, $X + Y = (1, 1, 1, 1, 1, ...)$, and thus the sequence converges as it's a constant sequence.

(b) their product XY converges.

Let $X = (0, 1, 0, 1, 0, 1, ...)$ and let $Y = (1, 0, 1, 0, 1, 0, ...)$. Once more, these two sequences clearly diverge as the difference of two consecutive terms is equal to 1.
Thus, $XY = (0, 0, 0, 0, 0, ...)$, which clearly converges.

3) Show that if X and Y are sequences such that X and $X + Y$ are convergent, then Y is convergent.

Recall Theorem 3.2.3:

Theorem 3.2.3. ***(a)*** *Let $X = (x_n)$ and let $Y = (y_n)$ be sequences of real numbers that converge to x and y, respectively, and let $c \in \mathbb{R}$. Then the sequences $X + Y$, $X - Y$, $X * Y$, and cX converge to $x + y$, $x - y$, xy, and cx, respectively.*

(b) *If $X = (x_n)$ converges to x and $Z = (z_n)$ is a sequence of nonzero real numbers that converges to z and $z \neq 0$, then the quotient sequence X/Z converges to x/z*

If X and $X+Y$ are convergent, then by Theorem 3.2.3, $Y = (X+Y) - X.$ is also convergent.

5) Show that the following sequence is not convergent.

   (a) (2^n)

   $2^n > n$, and (n) is an unbounded sequence. Therefore, (2^n) is also unbounded.

   Every convergent sequence must be bounded so we can conclude that (2^n) is unbounded.

6) Find the limits of the following sequence:

   (a) $\lim((\frac{2+1}{n})^2)$

$$\begin{aligned}
\lim\left(\left(2+\frac{1}{n}\right)^2\right) &= \lim\left(\left(2+\frac{1}{n}\right) * \left(2+\frac{1}{n}\right)\right) \\
&= \text{By Theorem 3.2.3 (a): Limit of a product = Product of limits} \\
&= \left(\lim\left(2+\frac{1}{n}\right)\right) * \left(\lim\left(2+\frac{1}{n}\right)\right) \\
&= \text{By Theorem 3.2.3 (a): Limit of a sum = Sum of Limits} \\
&= \left(\lim(2) + \lim\left(\frac{1}{n}\right)\right) * \left(\lim(2) + \lim\left(\frac{1}{n}\right)\right) \\
&= (2+0) * (2+0) \\
&= 4
\end{aligned}$$

9) Let $y_n := \sqrt{n_1} - \sqrt{n}$ for $n \in \mathbb{N}$. Show that $(\sqrt{n}y_n)$ converges. Find the limit.

To show that the sequences (y_n) and $(\sqrt{n}y_n)$ converge, we first need to recall Theorem 3.2.10:

Theorem 3.2.10. *Let $X = (x_n)$ be a sequence of real numbers that converges to x and suppose that $x_n \geq 0$. Then the sequence $(\sqrt{x_n})$ of positive square roots converges and* $\lim(\sqrt{x_n}) = \sqrt{x}$.

$$\begin{aligned}
y_n = \sqrt{n+1} - \sqrt{n} &= \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})} \\
&= \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} \\
&= \frac{1}{\sqrt{n+1} + \sqrt{n}}
\end{aligned}$$

$\implies \quad |y_n - 0| = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}}$. Now we have that $\lim(\frac{1}{n}) = 0$ implies $\lim(\frac{1}{\sqrt{n}}) = 0$ by Theorem 3.2.10.

Now, we have

$$\sqrt{n}y_n = \frac{\sqrt{n}}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{1+\sqrt{\frac{n+1}{n}}} = \frac{1}{1+\sqrt{1+\frac{1}{n}}}$$

Now, by the algebra of limits and of convergent sequences, we have that as $\lim(\frac{1}{n}) = 0 \implies \lim(1+\frac{1}{n}) = 1 \implies \lim\sqrt{1+\frac{1}{n}} = \sqrt{1} = 1$ and thus $\lim\left(\frac{1}{1+\sqrt{1+\frac{1}{n}}}\right) = \frac{1}{1+\sqrt{1+0}} = \frac{1}{2}$. Therefore $\lim(\sqrt{n}y_n)$ exists and is equal to $\frac{1}{2}$.

14) Use the Squeeze Theorem 3.2.7 to determine the limits of the following,

(a) (n^{1/n^2}).

Recall Theorem 3.2.7 Squeeze Theorem:

Squeeze Theorem. *Suppose that* $X = (x_n)$, $Y = (y_n)$, *and* $Z = (z_n)$ *are sequences of real numbers such that*

$$x_n \leq y_n \leq z_n \; \forall \; n \in \mathbb{N}$$

and that $\lim(x_n) = \lim(z_n)$. *Then* $Y = (y_n)$ *is convergent and*

$$\lim(x_n) = \lim(y_n) = \lim(z_n)$$

So, notice that

$$1 \leq n^{\frac{1}{n^2}} \leq n^{\frac{1}{n}}$$

and $\lim(n^{\frac{1}{n}}) = 1$. By the Squeeze Theorem,

$$1 \leq \lim(n^{\frac{1}{n^2}}) \leq \lim(n^{\frac{1}{n}}) = 1$$

Therefore we have that $\lim(n^{\frac{1}{n^2}} = 1)$.

16) Apply Theorem 3.2.11 to the following sequences, where a, b satisfy $0 < a < 1, b > 1$.

Recall Theorem 3.2.11:

Theorem 3.2.11. *Let* (x_n) *be a sequence of positive real numbers such that* $L := \lim(x_{n+1}/x_n)$ *exists. If* $L < 1$, *then* (x_n) *converges and* $\lim(x_n) = 0$.

(c) $(\frac{n}{b^n})$

Since $\frac{n}{b^n} > 0 \; \forall \; n$, we have that

$$\lim\left(\frac{\frac{n+1}{b^{n+1}}}{\frac{n}{b^n}}\right) = \frac{1}{b} < 1$$

Thus, let (x_n) be a sequence of positive real numbers such $L := \lim\left(\frac{x_{n+1}}{x_n}\right)$ exists. If $L < 1$, then (x_n) converges and $\lim(x_n) = 0$. Therefore we have that $\lim(\frac{n}{b^n}) = 0$.

(d) $(2^{3n}/3^{2n})$
Since $\frac{2^{3n}}{3^{2n}} > 0 \ \forall\ n \in \mathbb{N}$, we have that

$$\lim\left(\frac{\frac{2^{3(n+1)}}{3^{2(n+1)}}}{\frac{2^{3n}}{3^{2n}}}\right) = \frac{8}{9} < 1$$

Thus, let (x_n) be a sequence of positive real numbers such that $L := \lim(\frac{x_{n+1}}{x_n})$ exists. If $L < 1$, then (x_n) converges and $\lim(x_n) = 0$. Therefore $\lim(\frac{2^{3n}}{3^{2n}}) = 0$.

17) (a) Give an example of a convergent sequence (x_n) of positive numbers with $\lim(x_{n+1}/x_n) = 1$.

Let (x_n) be a sequence such that $x_n = 1 \ \forall\ n \in \mathbb{N}$. This is a constant sequence and is thus convergent and $\lim(\frac{x_{n+1}}{x_n}) = 1$.

(b) Give an example of a divergent sequence with this property. (Thus, this property cannot be used as a test for convergence.)

Let (x_n) be a sequence such that $(x_n) = (n)$. This sequence is divergent because it's not bounded, however $\lim(\frac{x_{n+1}}{x_n}) = 1$.

22) Suppose that if (x_n) is a convergent sequence and (y_n) is such that for any $\varepsilon > 0$ there exists M such that $|x_n - y_n| < \varepsilon$ for all $n \geq M$. Does it follow that (y_n) is convergent?

It does follow that (y_n) is convergent. In fact, $\lim(y_n) = \lim(x_n)$. To show this, let $x = \lim(x_n)$.

For any $\varepsilon > 0$, choose $M_1, M_2 > 0$ s.t.

$$|x_n - x| < \frac{\varepsilon}{2},\ \forall\ n \geq M_1 \qquad (a)$$

$$\text{and } |x_n - y_n| < \frac{\varepsilon}{2},\ \forall\ n \geq M_2 \qquad (b)$$

Choose $M = \max\{M_1, M_2\}$. Then $\forall\ n \geq M$,

$$\begin{aligned}
|y_n - x| &= |y_n - x_n + x_n - x| \\
&\leq |y_n - x_n| + |x_n - x| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon && \text{from (a) and (b)}
\end{aligned}$$

Thus we have that $\lim(y_n) = x$.

3. Give an example of each of the following:

(a) A convergent sequence of rational numbers having an irrational limit.

Let $a_n = \left(1 + \frac{1}{n}\right)^n$ be a sequence of rationals. Then, notice that $a_n \in \mathbb{Q}$, but notice also that $\lim_{n\to\infty} a_n = e$.

(b) A convergent sequence of irrational numbers having a rational limit.

Let $a_n = \frac{\sqrt{2}}{n}$. Then we have that $a_n \notin \mathbb{Q}$, but we also have that $\lim_{n\to\infty} a_n = 0$, and $0 \in \mathbb{Q}$

4. Prove: Let a_n and b_n be sequences of real numbers and $A \in \mathbb{R}$. If for some $k > 0$ and some $m \in \mathbb{N}$, we have $|a_n - A| \leq k|b_n|$ for all $n > m$, and if $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n = A$.

Proof. Let a_n and b_n be sequences of real numbers and let $A \in \mathbb{R}$. Let $k > 0$, and some $m \in \mathbb{N}$. We have $|a_n - A| \leq k|b_n| \ \forall\ n > m$, and $\lim_{n\to\infty} b_n = 0$. We want to show that $\lim_{n\to\infty} a_n = A$.

Let $\varepsilon > 0$ be given. Since $\lim b_n = 0$, we know that b_n converges. So by the definition of the limit of a sequence, we know that $\exists\ N_1$ s.t. $|b_n - 0| = |b_n| < \varepsilon \ \forall\ n \geq N_1$. This also means that since $\lim b_n = 0$, we have $k|b_n| < \varepsilon \implies |b_n| < \frac{\varepsilon}{k}$, by algebra.

Recall that we have $|a_n - A| \leq k|b_n| \ \forall\ n > m$. So, let $N = \max\{m, N_1\}$. Then we have that $\forall\ n \geq N,\ |a_n - A| \leq k|b_n| \leq k\left(\frac{\varepsilon}{k}\right) = \varepsilon$. Thus we have that a_n converges since it is also always less than ε. This is by the definition of the limit of a sequence.

$\therefore$ $\lim_{n\to\infty} a_n = A$; That is, the sequence a_n converges to A. ∎

5. (Similar to Section 3.2, problem 7)

(a) Suppose that $\lim_{n\to\infty} a_n = 0$. If b_n is a bounded sequence, prove that $\lim_{n\to\infty} a_n b_n = 0$.

Proof. Let b_n be a bounded sequence. Then by the definition of a bounded sequence we know that $\exists\ M > 0,\ M \in \mathbb{R}$ s.t. $|b_n| \leq M \ \forall\ n \in \mathbb{N}$.

Let $\varepsilon > 0$ be given. Since a_n converges, we know that by the definition of the limit of a sequence, $\exists\ N$ s.t. $|a_n - 0| = |a_n| < \frac{\varepsilon}{M} \ \forall\ n \geq N$, since we know that by the definition of bounded, $|b_n| \leq M$. Thus, by the utilization of the triangle inequality, we have that $|a_n b_n| \leq |a_n||b_n| \leq M\left(\frac{\varepsilon}{M}\right) = \varepsilon,\ \forall\ n \geq N$.

$\therefore$ we have that $a_n b_n$ converges to 0. ∎

(b) Show by counterexample that the boundedness of b_n is a necessary condition for part (a).

Let $a_n = \frac{1}{n}$, and let $b_n = n^3$. Then we have

$$\lim_{n\to\infty} a_n = 0, \text{ and } \lim_{n\to\infty} b_n = \infty$$

, since n^3 is an unbounded sequence since n^3 is not bounded above, and in order to be bounded, the sequence must be both bounded above and bounded below. So, we have the following:

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} \left(\frac{1}{n}\right)(n^3) = \lim_{n\to\infty} \left(\frac{n^3}{n}\right) = \lim_{n\to\infty} n^2 = \infty \neq 0$$

$\therefore$ the boundedness of b_n is a necessary condition for part (a) to be true.

6. Prove or justify, if true. Provide a counterexample, if false.

   (a) If a_n converges, then a_n/n also converges.

   This is true.

   Proof. Let $\lim a_n = A$. Let $b_n = n$, and let $\lim b_n = B$. Recall Theorem 3.2.3. Since $n \in \mathbb{N}$, we know that $n \neq 0$, and thus we know that $0 \notin b_n$. Thus by Theorem 3.2.3, we have that $\lim \frac{a_n}{b_n} = \frac{A}{B}$. And thus since the limit exists, we have that $\frac{a_n}{n}$ also converges. ■

   (b) If a_n does not converge, then a_n/n does not converge.

   This is false.

   Counterexample: Let $a_n = n$. Then we know that a_n does not converge, however $\frac{a_n}{n} = \frac{n}{n} = 1$, which converges since $\lim 1 = 1$.

   (c) If a_n converges and b_n is bounded, then $a_n b_n$ converges.

   This is false.

   Counterexample: Let $a_n = (\frac{1}{n}+1)$, and let $b_n = (-1)^n$. Then we have that $\lim a_n = 1$, and b_n is bounded both above and below, by -1 and 1, respectively. Thus $(-1)^n$ is bounded. So if we multiply the two together, we have

   $$a_n b_n = (\frac{1}{n} + 1)(-1)^n$$

   which is not convergent since this sequence oscillates. Thus by the fact that a convergent sequence multiplied by a divergent sequence diverges, we have that even though a_n is convergent and b_n is bounded, $a_n b_n$ does not converge.

   (d) If a_n converges to zero and $b_n > 0$ for all $n \in \mathbb{N}$, then $a_n b_n$ converges.

   This is false.

   Counterexample: Let $a_n = \frac{1}{n}$, and let $b_n = n^3$. Then we have that $\lim a_n = 0$, and that $b_n > 0 \; \forall \; n \in \mathbb{N}$. Then, we have that $a_n b_n = (\frac{1}{n})(n^3) = \frac{n^3}{n} = n^2$. And thus we have that $\lim a_n b_n = \lim n^2 = \infty$, thus $a_n b_n$ diverges.

(e) If $a_n \to A$ and $b_n \to A$ as $n \to \infty$, then $a_n = b_n$ for all $n \in \mathbb{N}$.

This is false.

Counterexample: Let $a_n = \frac{n}{3n+1}$, and let $b_n = \frac{1}{n} + 1$. Then we have $\lim a_n = 1$, and $\lim b_n = 1$, thus $\lim a_n = \lim b_n = 1$. However, $a_n \neq b_n$, since $\frac{n}{3n+1} \neq \frac{1}{n} + 1 = \frac{n}{n+1} \neq \frac{n}{3n+1}$.

(f) Every convergent sequence is bounded.

This is true.

Proof. Refer to proof of Theorem 3.2.2. ■

(g) Every bounded sequence is convergent.

This is false.

Counterexample: Let $a_n = (-1)^n$. Then we have that a_n is bounded both above and below, and thus a_n is bounded. However since a_n oscillates, we know that a_n is not convergent.

(h) If $a_n \to 0$ as $n \to \infty$, then for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n > N$, then $a_n < \epsilon$.

This is true.

Proof. Since we have that $\lim_{n\to\infty} a_n = 0$, then by the definition of the limit of a sequence, we have

$$\forall\ \varepsilon > 0,\ \exists\ N \in \mathbb{N} \text{ s.t. } |a_n - A| < \varepsilon,\ \forall\ n \geq N$$

Thus we have that

$$\begin{aligned} |a_n - 0| &< \varepsilon \\ |a_n| &< \varepsilon \\ -\varepsilon < a_n &< \varepsilon \end{aligned}$$

Thus we have that if $n > N$, then $a_n < \varepsilon$, by the definition of the limit of a sequence. ■

(i) If for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n > N$ implies $a_n < \epsilon$, then $a_n \to 0$ as $n \to \infty$.

This is false.

Counterexample: Let $a_n = -1,\ \forall\ n \in \mathbb{N}$. Then we have that by the definition of the limit of a sequence, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n > N$

implies that $a_n < \varepsilon$, but a_n does not converge to 0 as $n \to \infty$. Rather, we have that $a_n \to -1$.

(j) Given sequences a_n and b_n, if for some $A \in \mathbb{R}, k > 0$ and $m \in \mathbb{N}$ we have $|a_n - A| \leq k|b_n|$ for all $n > m$, then $a_n \to A$ as $n \to \infty$.

This is false.

Counterexample: Note the proof of question 4: We showed that the above statement is true if $\lim b_n = 0$ is also given. However, given that this limit is omitted, we have that we can provide a counterexample to disprove said statement.

Suppose $b_n = \frac{1}{n} + 1$. Then we have that $\lim b_n = 1$. Let $a_n = -2$. Assume for some $A \in \mathbb{R}, k > 0, m \in \mathbb{N}$, we have $|-2-(-2)| \leq k|1| \; \forall \; n > m = 2 \leq k$. However, note that there doesn't exist an m such that $2 \leq k \; \forall \; n > m$, since $m = n \; \forall \; m \in \mathbb{N}, \; n \in \mathbb{N}$.