Real Analysis Homework 12

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1. **(a)** **Section 6.2 Problem 15** Let I be an interval. Prove that if f is differentiable on I and if the derivative f' is bounded on I, then f satisfies a Lipschitz condition on I. (See Definition 5.4.4)

Proof. Let $M > 0$ be such that $|f'(c)| \leq M \; \forall \; c \in I$. This follows from the fact that f' is bounded on I. For $x, y \in I$ such that $x < y$, we know that by the mean value theorem, $\exists \; c \in (x, y) \subseteq I$ s.t. $f(y) - f(x) = f'(c)(y - x)$. This yields the following:

$$\begin{aligned} & |f(x) - f(y)| = |f'(c)||y - x| \leq M|x - y| \ldots (\because \; c \in I) \\ \implies & |f(x) - f(y)| \leq M|x - y| \; \forall \; x, y \in I \end{aligned}$$

This holds since $M > 0$ is true regardless of x, y. Thus we have that by the definition of a Lipschitz condition, f satisfies a Lipshitz condition on I.

■

(b) Suppose f and g are differentiable functions on (a, b). Show that between two consecutive roots of f there exists a root $f' + fg'$. (Hint: Apply Rolle's Theorem to the function $h(x) = f(x)e^{g(x)}$)

Proof. Let $x_1, x_2 \in (a, b)$ with $x_1 < x_2$, and $f(x_1) = f(x_2) = 0$.

Let $h(x) = f(x)^{g(x)}$.

Then we have $h(x_1) = h(x_2) = 0$ and h is differentiable on (a, b), and h is continuous on $[x_1, x_2]$.

Recall *Rolle's Theorem*:

Theorem 1 (**Rolle's Theorem**)**.** Suppose that f is continuous on a closed interval $I := [a, b]$, that the derivative f' exists at every point of the open interval (a, b), and that $f(a) = f(b) = 0$. Then there exists at least one point c in (a, b) such that $f'(c) = 0$.

Thus by *Rolle's Theorem*, we know that there exists some $c \in (x_1, x_2)$ s.t. $h'(c) = 0$ and

$$h'(x) = f'(x)e^{g(x)} + f(x) \cdot g'(x)e^{g(x)} = [f'(x) + f(x)g'(x)]e^{g(x)}$$

So $h'(c) = [f'(c) + f(c)g'(c)] \cdot e^{g(c)} = 0$

So $f'(c) + f(c)g'(c) = 0$. ■

(c) Suppose that f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b) = 0$. Prove that for each real number α, there exists some $c \in (a, b)$ such that $f'(c) = \alpha f(c)$. (Hint: Apply Rolle's Theorem to the function $g(x) = e^{-\alpha x} f(x)$)

Proof. Consider the function $g(x) = e^{-\alpha x} f(x)$. Since $f(a) = f(b) = 0$, we know that $g(a) = e^{-\alpha(a)} f(a) = e^{-\alpha(b)} f(b) = g(b)$, and thus $g(a) = e^{-\alpha(a)} \cdot 0 = 0 = g(b)$. By *Rolle's Theorem*, we know that since $f(a) = f(b) = 0$, f is continuous on $[a, b]$, and since f is differentiable on (a, b), then there exists some $c \in (a, b)$, such that $f'(c) = 0$. Thus there exists one point $c \in (a, b)$, such that $g'(c) = 0$, and

$$\begin{aligned} g'(x) &= -\alpha e^{-\alpha x} f(x) + e^{-\alpha x} f'(x) \\ 0 &= e^{-\alpha x} \left(-\alpha f(x) + f'(x)\right) \\ 0 &= -\alpha f(x) + f'(x) \\ \alpha f(x) &= f'(x) \end{aligned}$$

Thus we have that there exists some $c \in (a, b)$ such that $f'(c) = \alpha f(c)$. ■

(d) Suppose that f is differentiable on (a, b) and f' is bounded on (a, b). Show that f is uniformly continuous.

Proof. Let $M > 0$ satisfy $|f'(x)| \leq M \ \forall \ x \in (a, b)$. Recall the definition of differentiability:

Definition 1. Let $I \subseteq \mathbb{R}$ be an interval, let $f : I \to \mathbb{R}$, and let $c \in I$. We say that a real number L is the **derivative of f at c** if given any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $x \in I$ satisfies $0 < |x - c| < \delta(\varepsilon)$, then

$$\left| \frac{f(x) - f(c)}{x - c} - L \right| < \varepsilon.$$

In this case we say that f is **differentiable** at c, and we write $f'(c)$ for L. In other words, the derivative of f at c is given by the limit

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

provided this limit exists. (We allow the possibility that c may be the endpoint of the interval.)

Thus since f is differentiable on (a, b), we know that given any $\varepsilon > 0$, there exits $\delta(\varepsilon) > 0$ s.t. if $x \in (a, b)$ satisfies $0 < |x - c| < \delta(\varepsilon)$, then $\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$, for any $c \in (a, b)$.

By result of 1 (a), we know that since f is differentiable on (a, b), and since f' is bounded on (a, b), then f' satisfies a Lipschitz condition on (a, b).

Recall *Theorem 5.4.3*:

Theorem 2. If $f : A \to \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on A.

Thus we have that by *Theorem 5.4.3*, since f is a Lipschitz function, f is uniformly continuous on (a, b). ∎

(e) Give an example of a function f that is differentiable, uniformly continuous on (a, b), but f' is not bounded.

Consider the function $f : (0, \infty) \to (-\infty, \infty)$ given by $f(x) = x \sin\left(\frac{1}{x}\right)$. First, we must show that f is uniformly continuous, as follows:

$$\begin{aligned} |f(x) - f(c)| &= |x \sin\left(\frac{1}{x}\right) - c \sin\left(\frac{1}{c}\right)| \\ &\leq |x - c| && \because \max\left|\sin\left(\frac{1}{x}\right)\right| = 1, \text{ and } \max\left|\sin\left(\frac{1}{c}\right)\right| = 1 \\ &< \varepsilon \end{aligned}$$

Thus let $\delta(\varepsilon) = \varepsilon$. Hence f is uniformly continuous on $(-\infty, \infty)$. We also know that f is differentiable on $(-\infty, \infty)$ since

$$f'(x) = 1 \cdot \sin\left(\frac{1}{x}\right) + x \cdot \cos\left(\frac{1}{x}\right) \cdot -\frac{1}{x^2} = \sin\left(\frac{1}{x}\right) - \frac{x}{x^2} \cos\left(\frac{1}{x}\right) = \sin\left(\frac{1}{x}\right) - \frac{1}{x} \cos\left(\frac{1}{x}\right)$$

However, since the maximum value of sin and cos is 1, and since $x \in (1, \infty)$, the derivative $f'(x)$ is unbounded, since $\sup\{(0, \infty)\} = \infty$, and $\max\{(0, \infty)\} = \text{DNE}$, and $\lim_{x \to 0^+} \frac{1}{x} = \infty$. Thus $\frac{1}{x}$ can be infinitely large, meaning $f'(x)$ is unbounded.

2. Prove the given inequalities.

(a) $1 + x \leq e^x$ for all $x \in \mathbb{R}$.

Proof. Let $f : (0, \infty) \to \mathbb{R}$ given by $f(x) := e^x$. Then we know that $f'(x) = e^x$, which is greater than or equal to 1 since f is defined on $\mathbb{R}^+$. Then we know by the *Mean Value Theorem* that $\exists\, c \in (0, \infty)$ s.t. $f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{e^x - 1}{x} \implies e^x - 1 = f'(c) \cdot x \geq x$. Since $e^x \geq 1\ \forall\, x \in (0, \infty)$, we have that $e^x \geq x + 1\ \forall\, x \in (0, \infty)$.

For the case where $f : (-\infty, 0) \to \mathbb{R}$, we have that $f'(x) = e^x \leq 1\ \forall\, x \in (-\infty, 0)$. Now, by the *Mean Value Theorem*, we know that $\exists\, c \in (-\infty, 0)$ s.t. $f'(c) = \frac{f(0) - f(x)}{0 - x} = \frac{1 - e^x}{-x}$. Thus we have that $1 - e^x = f'(c) \cdot -x \leq 1$, which implies that $e^x - 1 = x \cdot -f'(c) \geq -1$. Thus $1 + x \leq e^x$.

∎

(b) $2x + 0.7 < e^x$ for all $x \geq 1$.

Proof. Let $f(x) = e^x - 2x - 0.7$. Then $f'(x) = e^x - 2 > 0 \; \forall \; x \geq 1$. So f is increasing on $[1, \infty)$. In particular, $f(x) = e^x - 2x - 0.7 \geq f(1) = e - 2.7 > 0$.

So $e^x > 2x + 0.7$. ∎

(c) $x^e \leq e^x$ for all $x > 0$.

Proof. Let $f(x) = x^{\frac{1}{x}} = y$. Then we have

$$\begin{aligned}
\ln(y) &= \frac{\ln(x)}{x} \\
\frac{1}{y} \cdot y' &= \frac{x\left(\frac{1}{x} - \ln(x)\right)}{x^2} \\
&= \frac{1 - \ln(x)}{x^2} \\
&\Downarrow \\
y' &= \frac{y(1 - \ln(x))}{x^2} \\
&= \frac{x^{\frac{1}{x}}(1 - \ln(x))}{x^2}
\end{aligned}$$

We note that $x = e$ is a critical point. Thus by the first derivative test, we know that f is increasing at e.

So $f(x) = x^{\frac{1}{x}} \leq e^{\frac{1}{x}} \; \forall \; x \in (0, \infty)$. Thus

$$\begin{aligned}
\frac{ln(x)}{x} &\leq \frac{1}{e} \\
\ln(x) &\leq \frac{x}{e} \\
(x &\leq e^{\frac{x}{e}})^e \\
x^e &\leq e^x \; \forall \; x > 0
\end{aligned}$$

∎

Section 6.3 - L'Hôpital's Rule

3. Section 6.3

6. (a) Evaluate $\lim\limits_{x \to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x}$

$$
\begin{aligned}
\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos(x)} &= \frac{e^0 + e^{-0} - 2}{1 - \cos(0)} \\
&= \frac{1 + 1 - 2}{1 - 1} \\
&= \frac{0}{0}
\end{aligned}
$$

We note that this is in one of the indeterminate forms that L'Hopital's Rule accounts for.

Recall L'Hopital's rules:

Theorem 3 (**L'Hopital's Rule, I**). Let $-\infty \le a < b \le \infty$ and let f, g be differentiable on (a, b) such that $g'(x) \neq 0$ for all $x \in (a, b)$. Suppose that

$$\lim_{x\to a+} f(x) = 0 = \lim_{x\to a+} g(x)$$

i. If $\lim_{x\to a+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$, then $\lim_{x\to a+} \frac{f(x)}{g(x)} = L$.

ii. If $\lim_{x\to a+} \frac{f'(x)}{g'(x)} = L \in \{-\infty, \infty\}$, then $\lim_{x\to a+} \frac{f(x)}{g(x)} = L$.

Theorem 4 (**L'Hopital's Rule, II**). Let $-\infty \le a < b \le \infty$ and let f, g be differentiable on (a, b) such that $g'(x) \neq 0$ for all $x \in (a, b)$. Suppose that

$$\lim_{x\to a+} g(x) = \pm\infty$$

i. If $\lim_{x\to a+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$, then $\lim_{x\to a+} \frac{f(x)}{g(x)} = L$.

ii. If $\lim_{x\to a+} \frac{f'(x)}{g'(x)} = L \in \{-\infty, \infty\}$, then $\lim_{x\to a+} \frac{f(x)}{g(x)} = L$.

Thus by utilizing L'Hopital's rule, we have

$$
\begin{aligned}
\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos(x)} &= \lim_{x\to 0^+} \frac{\frac{d}{dx}(e^x + e^{-x} - 2)}{\frac{d}{dx}(1 - \cos(x))} \\
&= \lim_{x\to 0^+} \frac{e^x - e^{-x}}{\sin(x)} \\
&= \frac{e^0 - e^{-0}}{\sin(0)} \\
&= \frac{1 - 1}{0} \\
&= \frac{0}{0}
\end{aligned}
$$

And since this is once again in one of the indeterminate forms that L'Hopital's rule accounts for, we perform the rule again, thus

$$\begin{aligned}
\lim_{x\to 0^+} \frac{e^x - e^{-x}}{\sin(x)} &= \lim_{x\to 0^+} \frac{\frac{d}{dx}(e^x - e^{-x})}{\frac{d}{dx}\sin(x)} \\
&= \lim_{x\to 0^+} \frac{e^x + e^{-x}}{\cos(x)} \\
&= \frac{e^0 + e^{-0}}{\cos(0)} \\
&= \frac{1+1}{1} \\
&= \frac{2}{1} \\
&= 2
\end{aligned}$$

Thus we have

$$\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1-\cos(x)} = \lim_{x\to 0^+} \frac{e^x - e^{-x}}{\sin(x)} = \lim_{x\to 0^+} \frac{e^x + e^{-x}}{\cos(x)} = 2$$

7. **(a)** Evaluate $\lim\limits_{x\to 0^+} \frac{\ln(x+1)}{\sin x}$ where the domain of the quotient is $(0, \pi/2)$.

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\ln(x+1)}{\sin(x)} &= \frac{\ln(0+1)}{\sin(0)} \\
&= \frac{\ln(1)}{0} \\
&= \frac{0}{0}
\end{aligned}$$

Since this is now in one of the indeterminate forms that L'Hopital's rule accounts for, we apply the rule as follows:

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\ln(x+1)}{\sin(x)} &= \lim_{x\to 0^+} \frac{\frac{d}{dx}\ln(x+1)}{\frac{d}{dx}\sin(x)} \\
&= \lim_{x\to 0^+} \frac{\frac{1}{x+1}\cdot 1}{\cos(x)} \\
&= \lim_{x\to 0^+} \frac{\frac{1}{x+1}}{\cos(x)} \\
&= \frac{\frac{1}{0+1}}{\cos(0)} \\
&= \frac{1}{1} \\
&= 1
\end{aligned}$$

Thus we have that

$$\lim_{x\to 0^+} \frac{\ln(x+1)}{\sin(x)} = \lim_{x\to 0^+} \frac{\frac{1}{x+1}}{\cos(x)} = 1$$

10. Evaluate the following limits:

(b) $\lim_{x\to 0}(1+3/x)^x \ (0, \infty)$

$$\begin{aligned}\lim_{x\to 0}\left(1+\frac{3}{x}\right)^x &= \left(1+\frac{3}{0}\right)^0 \\ &= (1+\infty)^0 \\ &= \infty^0\end{aligned}$$

Since this is in indeterminate form, we know that we need to somehow get it into the form in which it can be solved using L'Hopital's rule; So,

$$\begin{aligned}\lim_{x\to 0}\left(1+\frac{3}{x}\right)^x &= \lim_{x\to 0} e^{\ln\left(\left(1+\frac{3}{x}\right)^x\right)} \\ &= \lim_{x\to 0} e^{x\cdot\ln\left(1+\frac{3}{x}\right)} \\ &= e^{0\cdot\ln\left(1+\frac{3}{0}\right)} \\ &= 0\cdot 0\end{aligned}$$

and thus we can now use L'Hopital's rule on the exponent; So

$$\begin{aligned}&\lim_{x\to 0}\ln\left(1+\frac{3}{x}\right) \\ &= \lim_{x\to 0}\left[\frac{\ln(1+\frac{3}{x})}{\left(\frac{1}{x}\right)}\right] \\ &= \lim_{x\to 0}\left[\frac{\left(1+\frac{3}{x}\right)^{-1}\cdot\left(\frac{-3}{x^2}\right)}{\left(-\frac{1}{x^2}\right)}\right] \\ &= \lim_{x\to 0}\left[\frac{3}{1+\frac{3}{x}}\right] \\ &= 0\end{aligned}$$

And thus

$$\lim_{x\to 0}\left(1+\frac{3}{x}\right)^x = \lim_{x\to 0}\left[e^{x\ln\left(1+\frac{3}{x}\right)}\right] = e^0 = 1$$

(c) $\lim_{x\to\infty}(1+3/x)^x \ (0, \infty)$

We know from the previous problem that this will be in the indeterminate form of 1^∞, and thus we apply the same logic from step 2, we know that

$$\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x = \lim_{x\to 0}\left[e^{x\cdot\ln\left(1+\frac{3}{x}\right)}\right]$$

So

$$
\begin{aligned}
&\lim_{x\to\infty} \ln\left(1+\frac{3}{x}\right)\\
&= \lim_{x\to\infty}\left[\frac{\ln(1+\frac{3}{x})}{\left(\frac{1}{x}\right)}\right]\\
&= \lim_{x\to\infty}\left[\frac{\left(1+\frac{3}{x}\right)^{-1}\cdot\left(\frac{-3}{x^2}\right)}{\left(-\frac{1}{x^2}\right)}\right]\\
&= \lim_{x\to\infty}\left[\frac{3}{1+\frac{1}{x}}\right]\\
&= \frac{3}{1}\\
&= 3
\end{aligned}
$$

and thus

$$\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x = \lim_{x\to\infty}\left[e^{x\cdot\ln\left(1+\frac{3}{x}\right)}\right] = e^3$$

11. Evaluate the following limits:

(b) $\lim_{x\to 0^+}(\sin x)^x \ (0,\pi)$

We first note that

$$\lim_{x\to 0^+}(\sin(x))^x = (\sin(0))^0 = 0^0$$

So we know that we need to get this into a form that can utilize L'Hopital's rule, so we consider

$$(\sin(x))^x = e^{\ln((\sin(x))^x)} = e^{x\cdot\ln(\sin(x))}$$

This yields an indeterminate form of $0\cdot -\infty$, thus we can use L'Hopital's rule on the exponent; thus

$$
\begin{aligned}
&\lim_{x\to 0^+}\left[\frac{\ln(\sin(x))}{\frac{1}{x}}\right]\\
&= \lim_{x\to 0^+}\left[\frac{\frac{1}{\sin(x)}\cdot\cos(x)}{-\frac{1}{x^2}}\right]\\
&= \lim_{x\to 0^+}\left[\frac{-x^2}{\tan(x)}\right]\\
&= \lim_{x\to 0^+}\left[\frac{-2x}{\sec^2(x)}\right]\\
&= \frac{0}{1}\\
&= 0
\end{aligned}
$$

Which thus yields

$$\lim_{x\to 0}(\sin(x))^x = \lim_{x\to 0^+} e^{x\cdot\ln(\sin(x))} = e^0 = 1$$

(c) $\lim\limits_{x\to 0^+} x^{\sin x}$ $(0,\infty)$

We first note that
$$\lim_{x\to 0} x^{\sin(x)} = 0^{\sin(0)} = 0^0$$
which is in indeterminate form, thus we know that we need to somehow get this into an indeterminate form that can utilize L'Hopital's rule. Thus we consider
$$x^{\sin(x)} = e^{\ln\left(x^{\sin(x)}\right)} = e^{\sin(x)\cdot\ln(x)}$$

Thus we can use L'Hopital's rule on the exponent as follows:
$$\begin{aligned}
&\lim_{x\to 0^+}\left[\frac{\ln(x)}{\csc(x)}\right] \\
&= \lim_{x\to 0^+}\left[\frac{\frac{1}{x}}{-\csc(x)\cot(x)}\right] \\
&= \lim_{x\to 0^+}\left[\frac{-\cos(x)\tan(x) - \sin(x)\sec^2(x)}{1}\right] \\
&= 0
\end{aligned}$$

And thus we have that
$$\lim_{x\to 0^+} x^{\sin(x)} = \lim_{x\to 0^+} e^{\sin(x)\ln(x)} = e^0 = 1$$

4. Suppose that the function f is twice differentiable.

(a) Prove $f''(x) = \lim\limits_{h\to 0} \frac{f(x+3h)-3f(x+h)+2f(x)}{3h^2}$

Proof.

$$
\begin{aligned}
\lim_{h\to 0}\frac{f(x+3h)-3f(x+h)+2f(x)}{3h^2} &= \frac{f(x+3(0))-3f(x+0)+2f(x)}{3(0)^2}\\
&= \frac{f(x)-3f(x)+2f(x)}{0}\\
&= \frac{0}{0} \implies \text{ Use L'Hopital's Rule}\\
&\Downarrow\\
&= \lim_{h\to 0}\frac{3f'(x+3h)-3f'(x+h)}{6h}\\
&= \frac{3f'(x+3(0))-3f'(x+0)}{6(0)}\\
&= \frac{3f'(x)-3f'(x)}{0}\\
&= \frac{0}{0} \implies \text{ Use L'Hopital's Rule}\\
&\Downarrow\\
&= \lim_{h\to 0}\frac{9f''(x+3h)-3f''(x+h)}{6}\\
&= \frac{9f''(x+3(0))-3f''(x+(0))}{6}\\
&= \frac{9f''(x)-3f''(x)}{6}\\
&= \frac{6f''(x)}{6}\\
&= f''(x)
\end{aligned}
$$

Thus $f''(x) = \lim\limits_{h\to 0}\frac{f(x+3h)-3f(x+h)+2f(x)}{3h^2}$ ■

(b) Prove $f''(x) = \lim\limits_{h\to 0}\frac{2f(x+3h)-3f(x+2h)+f(x)}{3h^2}$

$$\begin{aligned}
\lim_{h\to 0}\frac{2f(x+3h)-3f(x+2h)+f(x)}{3h^2} &= \frac{2f(x+3(0))-3f(x+2(0))+f(x)}{3(0)^2}\\
&= \frac{2f(x)-3f(x)+f(x)}{0}\\
&= \frac{0}{0} \implies \text{ Use L'Hopital's Rule}\\
&\Downarrow\\
&= \lim_{h\to 0}\frac{6f'(x+3h)-6f'(x+2h)}{6h}\\
&= \lim_{h\to 0}\frac{f'(x+3h)-f'(x+2h)}{h}\\
&= \frac{f'(x+3(0))-f'(x+2(0))}{0}\\
&= \frac{f'(x)-f'(x)}{0}\\
&= \frac{0}{0} \implies \text{ Use L'Hopital's Rule}\\
&\Downarrow\\
&= \lim_{h\to 0}\frac{3f''(x+3h)-2f''(x+2h)}{1}\\
&= 3f''(x+3(0))-2f''(x+2(0))\\
&= 3f''(x)-2f''(x)\\
&= f''(x)
\end{aligned}$$

Thus $f''(x) = \lim\limits_{h\to 0}\frac{2f(x+3h)-3f(x+2h)+f(x)}{3h^2}$

5. Prove, if true or provide a counterexample, if false.

(a) If f and g are increasing on $[a,b]$, then $f+g$ is increasing on $[a,b]$.

This is a true statement.

Proof. Recall *Theorem 6.2.5*:

Theorem 5. Let $f : I \to \mathbb{R}$ be differentiable on the interval I. Then:

i. f is increasing on I if and only if $f'(x) \geq 0$ for all $x \in I$.

ii. f is decreasing on I if and only if $f'(x) \leq 0$ for all $x \in I$.

So we know that $f'(x) \geq 0\,\forall\, x \in [a,b]$, and $g'(x) \geq 0\,\forall\, x \in [a,b]$. Thus $f'(x)+g'(x) \geq 0\,\forall\, x \in [a,b]$, which implies that $f+g$ is increasing, by *Theorem 6.2.5*.

■

(b) If f and g are increasing on $[a,b]$, then fg is increasing on $[a,b]$.

This is a false statement. Consider $f, g : [-6, -4] \to \mathbb{R}$ given by $f(x) = 2x$, and $g(x) = 3x$. Then both $f'(x)$ and $g'(x)$ are greater than 0 for any $x \in [-6, -4]$, since $f'(x) = 2 \geq 0$ and $g'(x) = 3 \geq 0$. However, their product $fg = 2x \cdot 3x = 6x^2$, yet $fg'(x) \ngeq 0 \ \forall \ x \in [-6, -4]$, since $fg'(x) = 12x$, and $fg'(-6) = 12(-6) = -72 \ngeq 0$. Thus fg is not increasing.

(c) If f and g are differentiable on $[a, b]$ and $|f'(x)| \leq 1 \leq |g'(x)|$ for all $x \in (a, b)$, then $|f(x) - f(a)| \leq |g(x) - g(a)|$ for all $x \in [a, b]$.

This is a true statement.

Proof. There's two cases that we must consider: $f'(x) = g'(x)$, and $f'(x) \neq g'(x)$.

Case 1: Let $f'(x) = g'(x)$ such that $|f'(x)| = 1 = |g'(x)|$. Recall *Corollary 6.2.2*:

Corollary. Suppose that f and g are continuous on $I := [a, b]$, that they are differentiable on (a, b), and that $f'(x) = g'(x)$ for all $x \in (a, b)$. Then there exists a constant C such that $f = g + C$ on I.

Thus we have that $|f'(x)| \leq 1 \leq |g'(x)|$, and by *Corollary 6.2.2*, we know that there exists some constant C such that $g = f + C$. Thus

$$\begin{aligned}
|f(x) - f(a)| &\leq |g(x) - g(a)| \\
&\leq |(f(x) + C) - (f(a) + C)| && \text{By } \textit{Corollary 6.2.2} \\
&= |f(x) - f(a)|
\end{aligned}$$

Thus if $f'(x) = g'(x)$, and $|f'(x)| \leq 1 \leq |g'(x)| \ \forall \ x \in (a, b)$, then $|f(x) - f(a)| \leq |g(x) - g(a)|$ for all $x \in [a, b]$.

Case 2: Now, let $f'(x) \neq g'(x)$ such that $|f'(x)| < 1 < |g'(x)|$. Since f is differentiable on $[a, b]$, we know that the *Mean Value Theorem* holds true. Thus we have

$$\begin{aligned}
|f'(x)| &< 1 \\
\left|\frac{f(x) - f(c)}{x - c}\right| &< 1 \\
\frac{|f(x) - f(c)|}{|x - c|} &< 1 \\
|f(x) - f(c)| &< |x - c|
\end{aligned}$$

and

$$\begin{aligned}
1 &< |g'(x)| \\
1 &< \left|\frac{g(x) - g(c)}{x - c}\right| \\
1 &< \frac{|g(x) - g(c)}{|x - c|} \\
|x - c| &< |g(x) - g(c)|
\end{aligned}$$

Let $c = a$; then we have that

$$|f(x) - f(a)| < |x - a| < |g(x) - g(a)|$$

and thus

$$|f(x) - f(a)| < |g(x) - g(a)|$$

for all $x \in [a, b]$.

Hence we have that if $f'(x) \neq g'(x)$ satisfying $|f'(x)| < 1 < |g'(x)|$ for all $x \in (a, b)$, then $|f(x) - f(a)| < |g(x) - g(a)|$ for all $x \in [a, b]$.

Since both cases account for all possible outcomes of functions f and g whose derivatives satisfy the inequality $|f'(x)| \leq 1 \leq |g'(x)|\ \forall\ x \in (a, b)$, we have that this statement holds true. ∎

(d) A continuous function defined on a bounded interval assumes its maximum and minimum values.

This is a false statement. Consider the function $f : (-2, 2) \to (-6, 6)$ given by $f(x) = 3x$. Then we have that f is bounded on $(-2, 2)$, since $|f(x)| < 2\ \forall\ x \in (-2, 2)$. However, notice that $(f(-2), f(2)) = (-6, 6)$ is a bounded interval, yet $\min\{(-6, 6)\} = \text{DNE}$, and $\max\{(-6, 6)\} = \text{DNE}$, but $\inf\{(-6, 6)\} = -6$, and $\sup\{(-6, 6)\} = 6$. Thus f does not assume its minimum and maximum values since the minimum and maximum values do not exist.

(e) If f is continuous on $[a, b]$, then there exists a point $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

This is a false statement. Consider the function $f : \mathbb{R} \to (-1, 1)$ given by $f(x) := \sin\left(\frac{1}{x}\right)$. We know that f is not uniformly continuous, which thus means that f cannot be differentiable, hence it does not satisfy the *Mean Value Theorem.*

(f) Suppose f is differentiable on (a, b). If $c \in (a, b)$ and $f'(c) = 0$, then $f(c)$ is either the maximum or the minimum value of f on (a, b).

This is false. Consider the function $f : (-\infty, \infty) \to \mathbb{R}$ given by $f(x) = x^3$. Then we note that the derivative of $x^3 = 3x^2$. If we let $3x^2 = 0$, then we have that $x = 0$. However, the slope at $x = 0$ is 0, and thus we have that x is not a maximum or a minimum of f.

6. Not collected The following is an outline of a proof that e is irrational:

(a) Show that $f(x) = e^x$ is strictly increasing on $\mathbb{R}$.

(b) Use Taylor's Theorem about $x = 0$ and the estimate $e < 3$ to show for all $n \in \mathbb{N}$,

$$0 < e - \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right) < \frac{3}{(n+1)!}$$

(c) Suppose that e is rational. Then $e = a/b$ for some $a, b \in \mathbb{N}$. Choose $n > \max\{b, 3\}$. Substitute into part b) and show that this leads to the existence of an integer between 0 and $3/4$.