Real Analysis Homework 8

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1. Section 5.1

1. Prove the Sequential Criterion 5.1.3.

Recall the *Sequential Criterion*:

Theorem. A function $f : A \to \mathbb{R}$ is continuous at the point $c \in A$ if and only if for every sequence (x_n) in A that converges to c, the sequence $(f(x_n))$ converges to $f(c)$.

Let $\varepsilon > 0$ be given. Since we're given that f is continuous, we know that by the definition of continuity, there exists a $\delta > 0$ such that if $x \in A$ satisfies $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$. Thus, for that δ, since $x_n \to c$, $\exists\ n_0 \in \mathbb{N}$ s.t. $n \geq n_0 \implies |x_n - c| < \delta$. Thus we have

$$\begin{aligned} n \geq n_0 &\implies |x_n - c| < \delta \\ &\implies |f(x_n) - f(c)| < \varepsilon \\ &\implies \lim_{n\to\infty} f(x_n) = f(c) \end{aligned}$$

Now, suppose that for every sequence (x_n) in A that converges to c, $f(x_n)$ converges to $f(c)$. We will now show that the f is continuous at c.

Proof. We want to show that f is continuous at c.

By way of contradiction, suppose that f is discontinuous at c. Then we know that $\exists\ \varepsilon > 0\ \forall\ \delta > 0\ |x - c| < \delta$ and $|f(x) - f(c)| \geq \varepsilon$.

Let $\delta = \frac{1}{n}$. Then we have that x_n is such that $|x - c| < \frac{1}{n}$ and that $|f(x_n) - f(c)| \geq \varepsilon$.

$\therefore$ We have a sequence (x_n) such that $\lim\limits_{n\to\infty} x_n = c$ and $\lim\limits_{n\to\infty} f(x_n) \neq f(c)$. Notice however that we have a contradiction here, and thus we have that f is continuous at c. ■

5. Let f be defined for all $x \in \mathbb{R}, x \neq 2$, by $f(x) = (x^2 + x - 6)/(x - 2)$. Can f be defined at $x = 2$ in such a way that f is continuous at this point?

Since we have that f is not defined at $x = 2$, we know that we have to factor the numerator to evaluate the limit.

So, for $x \neq 2$:

$$f(x) = \frac{x^2 + x - 6}{x - 2} = \frac{\cancel{(x-2)}(x+3)}{\cancel{x-2}} = x + 3$$

Thus, if we let $f(2) := 5$, then f is continuous:

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x + 3) = 5$$

7. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at c and let $f(c) > 0$. Show that there exists a neighborhood $V_\delta(c)$ of c such that if $x \in V_\delta(c)$, then $f(x) > 0$.

Proof. Let $\varepsilon = \frac{f(c)}{2} > 0$. Since we are given that f is continuous at c, we know that there exists $\delta > 0$ such that

$$|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon = \frac{f(c)}{2}$$

Thus, if we define $V_\delta(c) := (c - \delta, c + \delta)$ and if we let $x \in V_\delta(c)$, then we have

$$\begin{aligned}
x \in V_\delta(c) &\implies |x - c| < \delta \\
&\implies |f(x) - f(c)| < \varepsilon \\
&\implies -\varepsilon < f(x) - f(c) < \varepsilon \\
&\implies f(c) - \frac{f(c)}{2} < f(x) && (\varepsilon = \frac{f(c)}{2}) \\
&\implies \frac{1}{2} f(c) < f(x) \\
&\implies f(x) > 0 && (f(c) > 0)
\end{aligned}$$

Thus we have that there exists a neighborhood $V_\delta(c)$ such that if $x \in V_\delta(c)$, then $f(x) > 0$. ■

11. Let $K > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy the condition $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in \mathbb{R}$. Show that f is continuous at every point $c \in \mathbb{R}$.

Proof. We want to show that f is continuous at every point $c \in \mathbb{R}$.

Let $\varepsilon > 0$ be given, and let $\delta = \frac{\varepsilon}{k}$. Then, for any $c \in \mathbb{R}$,

$$\begin{aligned}
|x - c| < \delta &\implies |x - c| < \frac{\varepsilon}{k} \\
&\implies k|x - c| < \varepsilon \\
&\implies |f(x) - f(c)| \leq k|x - c| < \varepsilon \\
&\implies |f(x) - f(c)| < \varepsilon
\end{aligned}$$

Thus we have that f is continuous for $c \in \mathbb{R}$. Since c is arbitrary, this also shows that f is continuous for all $c \in \mathbb{R}$. ■

12. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $f(r) = 0$ for every rational number r. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

Proof. We want to show that $f(x) = 0, \ \forall\, x \in \mathbb{R}$.

Let $x \in \mathbb{R}$. Since we know that $\mathbb{Q}$ is dense in $\mathbb{R}$, we know that we can find a sequence (x_n) of rational numbers such that (x_n) converges to x. Thus, by continuity of f at x, $(f(x_n))$ converges to $f(x)$. Since x_n is rational, we have that $f(x_n) = 0 \ \forall\, n \in \mathbb{N}$. Thus we have that $f(x) = \lim f(x_n) = \lim 0 = 0$.

$\therefore$ We have that $f(x) = 0$ for all $x \in \mathbb{R}$. ■

2. Use the definition of continuity to show that the given f is continuous.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$.

Recall the definition of continuity:

Definition 0.1. Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in A$. We say that f is **continuous at** c if, given any number $\varepsilon > 0$, there exists $\delta > 0$ such that if x is any point of A satisfying $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$. If f fails to be continuous at c, then we say that f is **discontinuous at** c.

Proof. In order to show that f is continuous for all $c \in \mathbb{R}$, we must consider two cases: $c \neq 0$, and $c = 0$.

Suppose $c \neq 0$. Then we have the following:

$$\begin{aligned}
|f(x) - f(c)| &= |x^2 - c^2| \\
&= |(x - c)(x + c)| \\
&= |x - c||x + c| \\
&< |x - c|(|x| + |c|) \\
&< \varepsilon
\end{aligned}$$

Which yields that

$$|x - c| < \frac{\varepsilon}{|x| + |c|} < \frac{\varepsilon}{|x|}$$

So, for the case when $c \neq 0$, let $\delta = \frac{\varepsilon}{|x|}$.

Now, suppose that $c = 0$. Then we have the following:

$$\begin{aligned} |f(x) - f(c)| &= |x^2 - 0^2| \\ &= |x^2| \\ &= (|x|)^2 \\ &< \varepsilon \end{aligned}$$

So, we have now that

$$(|x|)^2 < \varepsilon \implies |x| < \sqrt{\varepsilon}$$

Thus, let $\delta = \sqrt{\varepsilon}$.

$\therefore$ Since we have found definitions of δ that satisfy the definition of continuity, we have that f is continuous $\forall c \in \mathbb{R}$. ■

(b) Let $f : (0, \infty) \to \mathbb{R}$ be given by $f(x) = 1/x$.

Proof. Suppose $x > \frac{c}{2}$, since $c > 0$. Then we have that

$$|f(x) - f(c)| = \left|\frac{1}{x} - \frac{1}{c}\right| = \frac{|x-c|}{xc} < \frac{2}{c^2}|x-c| < \varepsilon \implies |x-c| < \frac{c^2\varepsilon}{2}$$

So, if we let $\delta < \min\{\frac{c^2\varepsilon}{2}, \frac{c}{2}\}$, then we have that if $|x - c| < \delta$, $x > \frac{c}{2}$, and $\left|\frac{1}{x} - \frac{1}{c}\right| < \frac{2}{c^2}\delta \leq \varepsilon$.

$\therefore$ We have that f is continuous $\forall\ c \in (0, \infty)$. ■

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = |x|$ (This is Problem 10 in Section 5.1)

Proof. We must first note the following:

$$\begin{aligned} |a| &= |a - b + b| \\ &\leq |a - b| + |b| \\ &\Downarrow \\ |a| - |b| &\leq |a - b| \end{aligned}$$

$\implies \pm(|a| - |b|) \leq |a - b|$ We also note:

$$\begin{aligned} |b| &= |b - a + a| \\ &\leq |b - a| + |a| \\ &= |a - b| + |a| \\ &\Downarrow \\ |b| - |a| &\leq |a - b| \end{aligned}$$

$$\implies ||a| - |b|| \leq |a - b|$$

$$\begin{aligned} |f(x) - f(c)| &= ||x| - |c|| \\ &\leq |x - c| \\ &< \varepsilon \end{aligned}$$

Thus if we let $\varepsilon = \delta$, then we have that $|x - c| < \delta$.

$\therefore$ We have that $f(x)$ is continuous $\forall\ c \in \mathbb{R}$. ■

3. Let $f(x) = \begin{cases} 3x + 2 & \text{if } x \text{ is rational} \\ 6 - x & \text{if } x \text{ is irrational} \end{cases}$

(a) Determine whether or not f is continuous at $x = 1$.

f is continuous at $x = 1$ since we proved in $HW7$ that $\lim\limits_{x \to 1} f(x)$ exists, $\lim\limits_{x \to 1} f(x) = f(1) = 5$.

(b) Determine whether or not f is continuous at $x = 0$.

f is not continuous for $x \neq 1$ (and not $x = 0$) since from Homework 7 again, $\lim\limits_{x \to c} f(x)$ does not exists for $x \neq 1$.

4. Let $g(x) = \begin{cases} 2x & \text{if } x \text{ is rational} \\ x + 3 & \text{if } x \text{ is irrational} \end{cases}$

Find all points where g is continuous (This is Problem 13 of Section 5.1)

First, let c be some point of continuity of g. Then since both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$, we can find sequences $(x_n) \subset \mathbb{Q}$ and $(y_n) \subset \mathbb{R} \setminus \mathbb{Q}$ such that $(x_n) \to c$ and $(y_n) \to c$. Then we have the following:

$$\begin{aligned} g(c) &= \lim_{n \to \infty} g(x_n) \\ &= \lim_{x \to \infty} 2x_n \\ &= 2c && (x_n \to c) \end{aligned}$$

$$\begin{aligned} g(c) &= \lim_{n \to \infty} g(y_n) \\ &= \lim_{n \to \infty} y_n + 3 \\ &= c + 3 && (y_n \to c) \end{aligned}$$

$$\begin{aligned} \implies 2c &= c + 3 \\ &\Updownarrow \\ c &= 3 \end{aligned}$$

Thus, we have that if g is continuous at a point c, then it must be $c = 3$. Thus we conjecture that g is continuous only at 3.

Proof.

$$\begin{aligned}
|g(x) - g(3)| &= |g(x) - 6| && (3 \in \mathbb{Q}) \\
&\leq \sup\{|2x - 6|, |(x + 3) - 6|\} && (x \text{ is either rational or irrational}) \\
&= \sup\{2|x - 3|, |x - 3|\} \\
&= 2|x - 3|
\end{aligned}$$

Thus $\forall\ \varepsilon > 0$ let $\delta = \frac{\varepsilon}{2}$. Then we have that

$$|x - 3| < \delta \implies |g(x) - g(3)| < \varepsilon$$

$\therefore$ We have that g is only continuous at $c = 3$. ■

5. Give an example of the following, if possible.

(a) A function f defined on $\mathbb{R}$ such that it is not continuous at any point in $\mathbb{R}$.

Consider the Dirichlet function. This function is not continuous at any points.

(b) A function f defined on $\mathbb{R}$ such that it is continuous at exactly one point in $\mathbb{R}$.

Consider the function $f(x) := \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

Notice that this function is continuous only at $x = 0$.

(c) A function f defined on $\mathbb{R}$ such that it is continuous at exactly two points in $\mathbb{R}$.

Consider the function $f(x) := \begin{cases} x^2, & \text{if } x \in \mathbb{Q} \\ 1, & \text{if } x \notin \mathbb{Q} \end{cases}$

We notice that this function is continuous only at the points -1 and 1.

6. Section 5.2

2. Show that if $f : A \to \mathbb{R}$ is continuous on $A \subseteq \mathbb{R}$ and if $n \in \mathbb{N}$, then the function f^n defined by $f^n(x) = (f(x))^n$, for $x \in A$, is continuous on A.

Proof. We want to show that f^n is continuous. We prove it by method of mathematical induction.

Basis Step: Let $n = 1$. Then we have that $f^1(x) = (f(x))^1 = f(x) \implies f^1 = f$. Since we are given that f is continuous, we know that f^1 is continuous as well.

Inductive Step: Assume that $f^n(x) = (f(x))^n$ is continuous $\forall\ n \in \mathbb{N}$.

Show: We want to show that $f^{n+1}(x) = (f(x))^{n+1}$ is continuous $\forall\ n \in \mathbb{N}$. Then we have

$$f^{n+1} = f^n \cdot f$$

Recall *Theorem 5.2.1*:

Theorem. Let $A \subseteq \mathbb{R}$, let f and g be functions on A to $\mathbb{R}$, and let $b \in \mathbb{R}$. Suppose that $c \in A$ and that f and g are continuous at c.

i. Then $f+g$, $f-g$, fg, and bf are continuous at c.

ii. If $h : A \to \mathbb{R}$ is continuous at $c \in A$ and if $h(x) \neq 0$ for all $x \in A$, then the quotient f/h is continuous at c.

Since both f^n and f are continuous functions, by *Theorem 5.2.1*, we have that $f^n \cdot f = f^{n+1}$ is a continuous function on A since it is the product of continuous functions on A.

$\therefore$ By the Principle of Mathematical Induction, we have that f^n is continuous $\forall\ n \in \mathbb{N}$ on A. $\blacksquare$

3. Give an example of functions f and g that are both discontinuous at a point c in $\mathbb{R}$ such that (a) the sum $f+g$ is continuous at c and (b) the product fg is continuous at c.

Consider the functions $f(x) := \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$, and $g(x) := \begin{cases} 0, & x = 0 \\ 1, & x \neq 0 \end{cases}$

We first note that both $f(x)$ and $g(x)$ are discontinuous at $x = 0$.

(a) Note that the sum

$$f(x) + g(x) := \begin{cases} 1+0, & x = 0 \\ 0+1, & x \neq 0 \end{cases} = \begin{cases} 1, & x = 0 \\ 1, & x \neq 0 \end{cases} = 1$$

. Thus, we have that the sum $f(x) + g(x)$ is a constant function, and thus is defined $\forall\ x \in \mathbb{R}$, and thus $f(x) + g(x)$ is continuous at c, where $c = 0$.

(b) Now, the product $f(x) \cdot g(x)$ is

$$f(x) \cdot g(x) := \begin{cases} 1 \cdot 0, & x = 0 \\ 0 \cdot 1, & x \neq 0 \end{cases} = \begin{cases} 0, & x = 0 \\ 0, & x \neq 0 \end{cases} = 0$$

Thus we have that the product of $f(x) \cdot g(x)$ is also a continuous function since $f(x) \cdot g(x) = 0$. Thus, we have that $f(x) \cdot g(x)$ is continuous $\forall\ x \in \mathbb{R}$. Hence $f(x) \cdot g(x)$ is continuous at c, where $c = 0$.

5. Let g be defined on $\mathbb{R}$ by $g(1) := 0$, and $g(x) := 2$ if $x \neq 1$, and let $f(x) := x + 1$ for all $x \in \mathbb{R}$. Show that $\lim_{x \to 0} g \circ f \neq (g \circ f)(0)$. Why doesn't this contradict Theorem 5.2.6?

Recall *Theorem 5.2.6*:

Theorem. Let $A, B \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be functions such that $f(A) \subseteq B$. If f is continuous at a point $c \in A$ and g is continuous at $b = f(c) \in B$, then the composition $g \circ f : A \to \mathbb{R}$ is continuous c.

Note that we have $g(x) := \begin{cases} 0, & x = 1 \\ 2, & x \neq 1 \end{cases}$, and $f(x) := x + 1$.

Thus $(g \circ f)(x) = g(f(x)) = g(x+1) = \begin{cases} 0, & x+1 = 1 \\ 2, & x+1 \neq 1 \end{cases} = \begin{cases} 0, & x = 0 \\ 2, & x \neq 0 \end{cases}$

This gives us that $\lim\limits_{x \to 0} (g \circ f)(x) = \lim\limits_{x \to 0} 2 = 2$, since $x \to 0 \implies x \neq 0$

Thus we have that $\lim\limits_{x \to 0} (g \circ f)(x) \neq (g \circ f)(0)$ since $2 \neq 0$.

The reason that this does not violate *Theorem 5.2.6* is because g is discontinuous at $f(0)$, since $f(0) = 1$.

7. Give an example of a function $f : [0,1] \to \mathbb{R}$ that is discontinuous at every point of $[0,1]$ but such that $|f|$ is continuous on $[0,1]$.

Let $f(x) := \begin{cases} -1, & x \in \mathbb{Q} \\ 1, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

Recall *Theorem 2.4.8 - The Density Theorem*:

Theorem (The Density Theorem). If x and y are any real numbers with $x < y$, then there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$.

And also recall *Corollary 2.4.9*:

Corollary. If x and y are real numbers with $x < y$, then there exists an irrational number z such that $x < z < y$.

Recall the *Discontinuity Criterion*:

Theorem (Discontinuity Criterion). Let $A \subseteq \mathbb{R}$, let $F : A \to \mathbb{R}$, and let $c \in A$. Then f is discontinuous at c if and only if there exists a sequence (x_n) in A such that (x_n) converges to c, but the sequence $(f(x_n))$ does not converge to $f(c)$.

Consider the sequence $x_n := (\frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3})$. Clearly $x_n \in [0,1]\ \forall\ n \in \mathbb{N}$. However, note that $\lim\limits_{x \to \frac{1}{3}} (x_n) = \frac{1}{3}$, but $\lim\limits_{x \to \frac{1}{3}} (f(x_n)) = 1$ but $f(\frac{1}{3}) = -1$. Thus we have that $\lim\limits_{x \to \frac{1}{3}} (f(x_n)) \neq f(c)$, where $c = \frac{1}{3}$. Thus by the *Discontinuity Criterion*, since there exists a sequence $(x_n) \in [0,1]$ such that $(x_n) \to c \in [0,1]$ but $(f(x_n)) \nrightarrow f(c)$. Thus we have that f is discontinuous at $c = \frac{1}{3}$. However, note that by the *Density Theorem* and by *Corollary 2.4.9*, we have that f is discontinuous at $c\ \forall c \in [0,1]$.

However, note that $|f(x)| = 1 \; \forall \; x \in [0,1]$. Thus we have that $|f|$ is continuous $\forall \; c \in [0,1]$.

8. Let f, g be continuous from $\mathbb{R}$ to $\mathbb{R}$ and suppose that $f(r) = g(r)$ for all rational numbers r. Is it true that $f(x) = g(x)$ for all $x \in \mathbb{R}$?

This is actually true.

Proof. Let $x \in \mathbb{R}$ be arbitrary. By the *Density Theorem*, we know that we can find a sequence $(x_n) \subset \mathbb{Q}$ s.t. $(x_n) \to x$.

Then we have:

$$\begin{aligned} f(x) &= \lim_{n\to\infty} f(x_n) && (f \text{ is continuous}) \\ &= \lim_{n\to\infty} g(x_n) && (x_n \in \mathbb{Q}) \\ &= g(x) && (g \text{ is continuous}) \\ \implies f(x) &= g(x) \; \forall \; x \in \mathbb{R} \end{aligned}$$

■

14. Let $g : \mathbb{R} \to \mathbb{R}$ satisfy the relation $g(x+y) = g(x)g(y)$ for all $x, y \in \mathbb{R}$. Show that if g is continuous at $x = 0$, then g is continuous at every point of $\mathbb{R}$. Also if we have $g(a) = 0$ for some $a \in \mathbb{R}$, then $g(x) = 0$ for all $x \in \mathbb{R}$.

Proof. We first note that since we're given that g is continuous at $x = 0$, we have

$$\begin{aligned} g(0) &= g(0+0) \\ &= g(0) \cdot g(0) && (g(x+y) = g(x)g(y), \; \forall x, y \in \mathbb{R}) \\ &= g(0)^2 \end{aligned}$$

This tells us that we have two cases to consider for $g(0)$: $g(0) = 0$ or $g(0) = 1$.

Case 1: $g(0) = 0$ Let $g(a) = 0$ for some $a \in \mathbb{R}$. Then we have that $\forall \; x \in \mathbb{R}$,

$$g(x) = g((x-a)+a) = g(x-a) \cdot g(a) = 0$$

Thus, if $g(a) = 0 \; \forall \; a \in \mathbb{R}$, then $g \equiv 0$ and thus g is constant. Thus, g is continuous $\forall \; x \in \mathbb{R}$.

Case 2: $g(0) = 1$ Let $g(c) \neq 0 \; \forall \; c \in \mathbb{R}$. Otherwise we have that g is a zero function, which was shown to be continuous as the result of *Case 1*, and this contradicts the fact that we let $g(0) = 1 \neq 0$.

Let $\varepsilon > 0$ be given, and let $c \in \mathbb{R}$ be arbitrary such that $c \neq 0$. Since g is continuous at 0 and $g(c) \neq 0$, we know that $\exists\ \delta > 0$ s.t. $\forall\ x \in \mathbb{R}$, we have:

$$|x - 0| < \delta \implies |g(x) - g(0)| < \frac{\varepsilon}{|g(c)|}$$

that is, since $g(0) = 1$,

$$|x| < \delta \implies |g(x) - 1| < \frac{\varepsilon}{|g(c)|}$$

Thus, for $x \in \mathbb{R}$ s.t. $|x - c| < \delta$, we have

$$\begin{aligned}
|g(x) - g(c)| &= |g((x-c)+c) - g(c)| \\
&= |g(x-c)g(c) - g(c)| && (g(a+b) = g(a)g(b)) \\
&= |g(c)(g(x-c) - 1)| \\
&= |g(c)||g(x-c) - 1| \\
&= |g(c)||g(h) - 1| && (\text{Substitue } h := x - c) \\
&< |g(c)|\frac{\varepsilon}{|g(c)|} && (|h| = |x - c| < \delta) \\
&= \varepsilon
\end{aligned}$$

$\therefore$ We have that g is continuous at c. Since $c \neq 0$ was arbitrary and g is continuous at 0, we have that g is continuous on $\mathbb{R}$. ■

7. Prove or justify if true. Provide a counterexample if false.

(a) Let f be defined on $[a, b]$. Let x_n be any sequence in $[a, b]$. The sequence $\{f(x_n)\}$ converges to $f(c)$ for x_n converging to $c \in [a, b]$ implies that f is continuous at $x = c$.

This is true. Recall the *Sequential Criterion for Continuity*:

Theorem (**Sequential Criterion for Continuity**)**.** A function $f : A \to \mathbb{R}$ is continuous at the point $c \in A$ if and only if for every sequence (x_n) in A that converges to c, the sequence $(f(x_n))$ converges to $f(c)$.

This is true by the *Sequential Criterion for Continuity.*

(b) If f is continuous on D and the sequence x_n in D is a converging sequence, then the sequence $\{f(x_n)\}$ converges.

This statement is false. Consider the function $f : (0, 1) \to \mathbb{Q}$ given by $f(x) = \frac{1}{x}$, and consider the sequence $x_n := (0.1, 0.01, 0.001, 0.0001, \dots)$, $\forall\ n \in \mathbb{N}$. Then we have that $x_n \to 0$, but since $0 \notin (0, 1)$ and since $(0, 1)$ is not a compact set, this violates the *Sequential Criterion for Continuity* and thus this statement is false.