Real Analysis Homework 7

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1. Section 4.1

1. Determine a condition on $|x-1|$ that will assure that:

(a) $|x^2-1| < \frac{1}{2}$
We notice first that $|x^2-1| = |x-1||x+1|$.

Consider the case where $|x-1| < 1$. Then we have that $-1 < x-1 < 1$ and thus $-3 < 1 < x+1 < 3$ and thus $|x+1| < 3$.

Hence we have that $|x^2-1| = |(x-1)(x+1)| = |x-1||x+1| < 1 \cdot 3 = 3$.

Now, since $\frac{1}{6} < 1$, if $|x-1| < \frac{1}{6}$, then $|x+1| < 3$ and thus $|x^2-1| = |(x-1)(x+1)| = |x-1||x+1| < \frac{1}{6} \cdot 3 = \frac{1}{2}$.

$\therefore$ If we let $|x-1| < \frac{1}{6}$ then $|x^2-1| < \frac{1}{2}$.

(c) $|x^2-1| < 1/n$ for a given $n \in \mathbb{N}$.

Notice that $0 < \frac{1}{n} \leq 1$, and thus $0 < \frac{1}{3n} \leq \frac{1}{3} < 1$.

So, by *Part (a)*, we have that if $|x-1| < \frac{1}{3n}$, then $|x+1| < 3$. Hence, $|x^2-1| = |(x-1)(x+1)| = |x-1||x+1| < \frac{1}{3n} \cdot 3 = \frac{1}{n}$.

$\therefore$ If $|x-1| < \frac{1}{3n}$, then $|x^2-1| < \frac{1}{n}$.

7. Show that $\lim\limits_{x \to c} x^3 = c^3$ for any $c \in \mathbb{R}$. (Hint: Use $c = 2$)

Suppose $|x-2| < 1$.

$$|x^3-8| = |x-2||x^2+2x+4|$$

So, we have

$$
\begin{aligned}
|x^2 + 2x + 4| = |(x^2 - 4x + 4) + 6x| \\
&= |(x-2)^2 + 6(x-2) + 12| \\
&\le |x-2|^2 + 6|x-2| + 12 \\
&< 1 + 6 + 12 = 19
\end{aligned}
$$

So we have that if we let $\delta = \min\{1, \frac{\varepsilon}{19}\}$, then we have that $\lim\limits_{x \to c} x^3 = c^3$, for $c = 2$.

More generally, we have the following:

$\forall\ c \in \mathbb{R}$, let $b = |c| + 1$. Then we have that if $|x| < b$, then

$$|x^2 + cx + c^2| \le |x|^2 + |c||x| + |c|^2 \le 3b^2$$

Now, if we let $\varepsilon > 0$ be arbitrary, we have that for $\delta = \frac{\varepsilon}{3b^2}$ and x such that $|x - c| < \delta$, then

$$
\begin{aligned}
|x^3 - c^3| &= |(x-c)(x^2 + cx + c^2)| \\
&\le |x-c||x^2 + cx + c^2| \\
&\le 3b^2|x-c| \\
&< 3b^2 \frac{\varepsilon}{3b^2} \\
&= \varepsilon
\end{aligned}
$$

Hence we have that $\lim\limits_{x \to c} x^3 = c^3,\ \forall\ c \in \mathbb{R}$.

9. Use either the $\varepsilon - \delta$ definition of the limit or the Sequential Criterion for limits, to establish the following limit:

(b) $\lim\limits_{x \to 1} \frac{x}{1+x} = \frac{1}{2}$

Let $\varepsilon > 0,\ \delta = \min\{\frac{1}{2}, \varepsilon\}$. Now for $0 < |x - 1| < \delta \le \frac{1}{2}$, we have:

$$
\begin{aligned}
\left|\frac{x}{1+x} - \frac{1}{2}\right| &= \left|\frac{2x - (1+x)}{2(1+x)}\right| \\
&= \left|\frac{x-1}{2(1+x)}\right| \\
&< |x-1| \qquad\qquad (|x-1| < \frac{1}{2} \implies x+1 > \frac{3}{2}) \\
&< \delta \\
&< \varepsilon
\end{aligned}
$$

And thus by the $\varepsilon - \delta$ definition of the limit, we have

$$\lim_{x \to 1} \frac{x}{1+x} = \frac{1}{2}$$

10. Use the definition of a limit to show that

(a) $\lim_{x\to 2}(x^2 + 4x) = 12$

$$\begin{aligned}
|x-2| < 1 \implies |x^2+4x-12| &\leq |(x+6)(x-2)| \\
&\leq |x+6||x-2| \\
&= |x-2+8||x-2| \\
&\leq (|x-2|+8)|x-2| \\
&\leq (1+8)|x-2| \qquad\qquad (|x-2|<1) \\
&= 9|x-2|
\end{aligned}$$

For $\delta = \min\{1, \frac{\varepsilon}{9}\}$ and x such that $|x-2| < \delta$ we have that $|x^2+4x-12| < \varepsilon$.

$\therefore \lim_{x\to 2} x^2 + 4x = 12.$

(b) $\lim_{x\to -1} \frac{x+5}{2x+3} = 4$

$$\begin{aligned}
\left|\frac{x+5}{2x+3} - 4\right| &= \left|\frac{x+5-4(2x+3)}{2x+3}\right| \\
&= \left|\frac{-7x-7}{2x+3}\right| \\
&= 7\left|\frac{x+1}{2x+3}\right| \\
&= \frac{7|x+1|}{|2x+3|}
\end{aligned}$$

Now, if $|x+1| < \frac{1}{4}$, then

$$\begin{aligned}
-\frac{5}{4} < x < -\frac{3}{4} &\implies \frac{1}{2} < 2x+3 < \frac{3}{2} \\
&\implies 0 < \frac{1}{2x+3} < 2 \\
\implies \left|\frac{x+5}{2x+3} - 4\right| &\leq \frac{7|x+1|}{|2x+3|} \\
&< 14|x+1|
\end{aligned}$$

Thus, let $\delta := \min\{\frac{1}{4}, \frac{\varepsilon}{14}\}$.

$\therefore \lim_{x\to -1} \frac{x+5}{2x+3} = 4$

15. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by the setting $f(x) := x$ if x is rational, and $f(x) = 0$ if x is irrational.

(a) Show that f has a limit at $x = 0$.

$\forall\ \varepsilon > 0$, choose $\delta = \varepsilon$. Then we have that $\forall\ x \in \mathbb{R}$ with $|x| < \delta$, we have

$$|f(x) - 0| = |f(x)| = \begin{cases} |x|, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

$$\implies |f(x)| < \varepsilon$$

$\therefore$ We have that f has a limit of 0 at $x = 0$.

(b) Use a sequential argument to show that if $c \neq 0$, then f does not have a limit at c.

Let $\{x_n\}$ and $\{y_n\}$ be two sequences, both converging to c s.t. $x_n \in \mathbb{R}\backslash\mathbb{Q}$ and since the rationals and irrationals are dense in the set of real numbers, $y_n \in \mathbb{Q}\ \forall\ n \in \mathbb{N}$.

Thus, $f(x_n) = 0\ \forall\ n$ and $f(y_n) = y_n\ \forall\ n$. This yields that $f(x_n) \to 0$, but $f(y_n) = y_n \to c$. Thus we now have that $\lim\limits_{n\to\infty} f(x_n) \neq \lim\limits_{n\to\infty} f(y_n)$. Hence we have that f does not have a limit at $x = c$.

2. Use the definition of a limit to establish the following limits:

(a) $\lim\limits_{x\to 1} \frac{x^2-x-2}{2x-3} = 2$

We want to show the following $\forall\ \varepsilon > 0$:

$$\begin{aligned} \left|\frac{x^2 - x - 2}{2x - 3} - 2\right| &= \left|\frac{x^2 - x - 2 - 4x + 6}{2x - 3}\right| \\ &= \left|\frac{x^2 - 5x + 4}{2x - 3}\right| \\ &= \frac{|x - 4||x - 1|}{|2x - 3|} \\ &< \varepsilon \end{aligned}$$

So, we have for the numerator:

$$\begin{aligned} |x - 4| &= |(x - 1) - 3| \\ &= |x - 1| - 3 \\ &\leq 1 - 3 \\ &= -2 \end{aligned}$$

And for the denominator we have for $|x-1|$:

$$\begin{aligned}-1 < x-1 &< 1\\ 0 < x &< 2\\ 0 < 2x &< 4\\ -3 < 2x-3 &< 1\\ -\frac{1}{3} &> \frac{1}{2x-3}\\ \frac{1}{2x-3} &< -\frac{1}{3}\end{aligned}$$

This then yields that

$$\frac{|x-4||x-1|}{|2x-3|} < (-2)\cdot\left(-\frac{1}{3}\right)|x-1| < \varepsilon$$

Which then gives us that

$$|x-1| < \frac{3\varepsilon}{2}$$

Hence if we let $\delta = \min\{1, \frac{3\varepsilon}{2}\}$, and x such that $|x-1| < \delta$ gives us that $\left|\frac{x^2-x-2}{2x-3} - 2\right| = \frac{|x-4||x-1|}{|2x-3|} < \varepsilon,\ \forall\ \varepsilon > 0$.

$\therefore\ \lim\limits_{x\to 1} \frac{x^2-x-2}{2x-3} = 2$

(b) $\lim\limits_{x\to 5}(x^2-3x+1) = 11$

We want to show the following $\forall\ \varepsilon > 0$:

$$\begin{aligned}|x^2-3x+1-11| &= |x^2-3x-10|\\ &= |x-5||x+2|\\ &< \varepsilon\end{aligned}$$

So we have the following for $|x+2|$:

$$\begin{aligned}|x+2| &= |(x-5)+7|\\ &= |x-5|+7\\ &\leq 1+7\\ &= 8\end{aligned}$$

This then yields that $|x-5||x+2| < 8|x-5| = \varepsilon$, and thus $|x-5| < \frac{\varepsilon}{8}$.

So if we let $\delta = \min\{1, \frac{\varepsilon}{8}\}$, and x such that $|x-5| < \delta$, we have that $|x^2-3x-1| = |x-5||x+2| < \varepsilon$.

$\therefore\ \lim\limits_{x\to 5}(x^2-3x+1) = 11$

3. Find a $\delta > 0$ so that $|x-2| < \delta$ implies that

(a) $|x^2 + x - 6| < 1$

Note that $|x^2+x-6| = |x-2||x+3|$. So, we have that $|x+3| = |(x-2)+5| = |x-2|+5 \leq 1+5 = 6$. Thus we have that $6|x-2| < 1 \implies |x-2| < \frac{1}{6}$. So if we let $\delta = \min\{1, \frac{1}{6}\} = \frac{1}{6}$ gives us that $|x-2| < \delta$.

(b) $|x^2 + x - 6| < 1/n$ for a given $n \in \mathbb{N}$

By similar logic as the previous problem, we have that if we let $\delta = \min\{1, \frac{1}{6n}\} = \frac{1}{6n}$, we have that $|x-2| < \delta$.

(c) $|x^2 + x - 6| < \varepsilon$

Suppose that $|x-2| < 1$. Then we have that $|x+3| = |(x-2)+5| \leq |x-2|+5 < 6$. So we have that if we let $\delta = \min\{1, \frac{\varepsilon}{6}\}$, we have that $|x-2| < \delta$.

4. Show that for any a, $\lim\limits_{x \to a} F(x)$ does not exist for $F(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$

Let x_n = sequence of rationals $\to a$, and let y_n = sequence of irrationals such that $\to a$. Then we have that $\lim f(x_n) = 1$ and $\lim f(y_n) = 0$. Thus by *Theorem 4.1.9*, we have that $\lim f(x)$ does not exist.

5. Let $f(x) = \begin{cases} 3x+2 & \text{if } x \text{ is rational} \\ 6-x & \text{if } x \text{ is irrational} \end{cases}$

(a) Show $\lim\limits_{x \to 1} f(x) = 5$

Let $\varepsilon > 0$ be given. Then we have the following for $|f(x) - L|$:

$$\begin{aligned} |f(x) - 5| &= \begin{cases} |(3x+2) - 5| \\ |(6-x) - 5| \end{cases} \\ &= \begin{cases} |3x-3| \\ |1-x| \end{cases} \\ &= \begin{cases} 3|x-1| \\ |x-1| \end{cases} \\ &= \varepsilon \end{aligned}$$

Thus, let $\delta = \min\{\varepsilon, \frac{\varepsilon}{3}\}$, and we have that $\lim\limits_{x \to 1} f(x) = 5$.

(b) Show $\lim\limits_{x \to a} f(x)$ does not exist if $a \neq 1$.

Let x_n be a sequence of rational numbers such that $x_n \neq a$, $\forall\, n \in \mathbb{N}$, and $\lim(x_n) = a$. Let y_n be a sequence of irrational numbers such that $y_n \neq a$, $\forall\, n \in \mathbb{N}$, and $\lim(y_n) = a$. Then we have that $\lim\limits_{x\to a} f(x_n) = 3a + 2$, and that $\lim\limits_{x\to a} f(y_n) = 6 - a$. Thus, we have that if the limit exists, then $\lim x \to a f(x_n) = \lim\limits_{x\to a} f(y_n)$. So,

$$\begin{aligned} 3a + 2 &= 6 - a \\ 4a + 2 &= 6 \\ 4a &= 4 \\ a &= 1 \end{aligned}$$

However, since we have that $a \neq 1$, we have that by *Theorem 4.1.8* (*The Divergence Criteria*), we know that $3a + 2 \neq 6 - a \implies \lim\limits_{x\to a} f(x_n) \neq \lim\limits_{x\to a} f(y_n)$, and thus this limit does not exists if $a \neq 1$.

6. Let $f : [-1, 1] \to \mathbb{R}$ be given by, $f(x) = \begin{cases} 0, & \text{if } x = \pm\frac{1}{n} \text{ where } n \in \mathbb{N} \\ 1, & \text{otherwise} \end{cases}$

Find the limit, if it exists, using the definition of a limit.

(a) $\lim\limits_{x\to\frac{3}{8}} f(x)$

Notice first that the range of f is $R(f) := \{0, 1\}$. This yields that $\varepsilon = 0$, or $\varepsilon = 1$. However, by the definition of a limit, we have that $\varepsilon > 0$, and thus $\varepsilon = 1$. This gives us a guarantee that ε will always equal 1. Thus all we need to worry about is finding a δ, that doesn't need to be in terms of ε.

First, recall the definition of a limit when a function $f : A \to \mathbb{R}$ for $A \subseteq \mathbb{R}$, and c is a cluster point of A:

$$\lim_{x\to c} f(x) = L \implies \forall\, \varepsilon > 0\ \exists\, \delta > 0 \text{ s.t. } x \in A \land 0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$$

However, also recall the definition of a cluster point of A for $c \in \mathbb{R}$:

$$\forall\, \delta > 0\ \exists\, x \in V_\delta(c) := \{x \in A : |x - c| < \delta\} = (c - \delta, c + \delta) \text{ s.t. } x \neq c$$

So we have now that if we observe the different parts of the piecewise function that create the domain are the following sets: $B := \{-1, -\frac{1}{2}, -\frac{1}{3}, \ldots, \frac{1}{3}, \frac{1}{2}, 1\}$ and $C := [-1, 1] \setminus B$.

We now note that since $c = \frac{3}{8}$, we know by evaluating the limit through substitution, $\lim\limits_{x\to\frac{3}{8}} = 1$. Thus we also know that since $\frac{3}{8}$ is not of the form $\pm\frac{1}{n}$ for $n \in \mathbb{N}$, that by the corollaries to *The Archimedian Property*, there exists a rational number between any two real numbers. Thus we have that there must exist two rational numbers $p, q \in \mathbb{Q}$ s.t. $p < \frac{3}{8} < q$.

We note by the elements of B that if we let $p = \frac{1}{3}$ and let $q = \frac{1}{2}$ we have that $\frac{1}{3} < \frac{3}{8} < \frac{1}{2}$. So we now have that since δ must be defined such that $0 < |x - c| < \delta$, we can solve for δ as being $\frac{3}{8} - \frac{1}{3} = \frac{1}{24}$, since $\frac{1}{3} < \frac{1}{2}$, and thus we have that if we let $\delta = \frac{1}{24}$, then $(c - \delta, c + \delta)$ is such that $|x - c| < \delta$.

$\therefore \lim_{x \to \frac{3}{8}} f(x) = 1$ when $|x - \frac{3}{8}| < \delta$ for $\delta = \frac{1}{24}$

(b) $\lim_{x \to -\frac{1}{3}} f(x)$

By applying similar logic as the previous problem, we notice that since $-\frac{1}{2} < -\frac{1}{3} < -\frac{1}{4}$, since δ does not need to be defined in terms of ε since $\varepsilon = 1$, we have that we can let $\delta = |-\frac{1}{3} + \frac{1}{4}| = \frac{1}{12}$, since $\delta > 0$. Thus, we have that the cluster point $-\frac{1}{3}$ is defined such that $|x + \frac{1}{3}| < \frac{1}{12}$. Hence $\lim_{x \to -\frac{1}{3}} f(x) = 0$.

(c) $\lim_{x \to 0} f(x)$

We have that the limit does not exist when $x = 0$. We can see this if we let $x_n = \frac{1}{n} \to 0$, and $y_n = \frac{\sqrt{2}}{n} \to 0$. Then we have that $f(x_n) = 0$ and $f(y_n) = 1$. Thus we have that $\lim_{x \to 0} f(x)$ does not exist.

7. Section 4.2

3. Find $\lim_{x \to 0} \frac{\sqrt{1+2x} - \sqrt{1+3x}}{x + 2x^2}$ where $x > 0$.

Recall *Theorem 4.2.4*:

Theorem. let $A \subseteq \mathbb{R}$, let f and g be functions on A to $\mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. Further, let $b \in \mathbb{R}$.

i. If $\lim_{x \to c} f = L$ and $\lim_{x \to c} g = M$, then

$$\lim_{x \to c}(f + g) = L + M,$$
$$\lim_{x \to c}(f - g) = L - M,$$
$$\lim_{x \to c}(fg) = LM,$$
$$\lim_{x \to c}(bf) = bL.$$

ii. If $h : A \to \mathbb{R}$, if $h(x) \neq 0$ for all $x \in A$, and if $\lim_{x \to c} h = H \neq 0$, then

$$\lim_{x \to c} \left(\frac{f}{h} \right) = \frac{L}{H}$$

This yields the following:

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{1+2x}-\sqrt{1+3x}}{x+2x^2} &= \lim_{x\to 0}\frac{(1+2x)-(1+3x)}{(x+2x^2)(\sqrt{1+2x}+\sqrt{1+3x})} \\
&= \lim_{x\to 0}\frac{-x}{x(1+2x)(\sqrt{1+2x}+\sqrt{1+3x})} \\
&= \lim_{x\to 0}\frac{-1}{(1+2x)(\sqrt{1+2x}+\sqrt{1+3x})} \\
&= \frac{-1}{(1+0)(\sqrt{1+0}+\sqrt{1+0})} && \text{by } \textit{Theorem 4.2.4} \\
&= \frac{-1}{1(2)} \\
&= \frac{-1}{2}
\end{aligned}$$

Thus we have that $\lim\limits_{x\to 0}\frac{\sqrt{1+2x}-\sqrt{1+3x}}{x+2x^2} = -\frac{1}{2}$

4. Prove that $\lim\limits_{x\to 0}\cos(1/x)$ does not exist but that $\lim\limits_{x\to 0}x\cos(1/x)=0$

Proof. Let $x_n := \left((2n+1)\frac{\pi}{2}\right)^{-1} \implies x_n \to 0$. And also let $y_n := (2n\pi)^{-1}$. Then we have that $y_n \to 0$. We can now note that $\cos\left(\frac{1}{x_n}\right) = \cos\left((2n+1)\frac{\pi}{2}\right) = 0$, and that $\cos\left(\frac{1}{y_n}\right) = \cos(2n\pi) = 1$. This yields that $\cos\left(\frac{1}{x_n}\right) \to 0$ and $\cos\left(\frac{1}{y_n}\right) \to 1$.

Recall the *Sequential Criterion*:

Theorem (Sequential Criterion)**.** Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. Then the following are equivalent.

i. $\lim\limits_{x\to c} f = L$.

ii. For every sequence (x_n) in A that converges to c such that $x_n \neq c$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to L.

So we have that $\lim\limits_{x\to 0}\cos\left(\frac{1}{x}\right)$ doesn't exist. However, $\lim\limits_{x\to 0}x\cos\left(\frac{1}{x}\right) = 0$. This is given to us by the fact that $\left|x\cos\left(\frac{1}{x}\right)\right| \le |x|$. Thus, if we let $\delta = \varepsilon$, we have that $\lim\limits_{x\to 0}x\cos\left(\frac{1}{x}\right) = 0$. ∎

6. Use the definition of the limit to prove the first assertion in Theorem 4.2.4(a).

Proof. Let $\varepsilon > 0$ be given.

Since we have that $\lim\limits_{x\to c} f(x) = L$, by the definition of the limit, we know that $\exists\, \delta_f > 0$ s.t. $|x-c| < \delta_f \implies |f(x)-L| < \frac{\varepsilon}{2}$.

Since $\lim_{x \to c} g(x) = M$, then we have that by the definition of the limit, we know that $\exists\ \delta_g > 0$ s.t. $|x - c| < \delta_g \implies |g(x) - M| < \frac{\varepsilon}{2}$.

Now, define $\delta = \max\{\delta_f, \delta_g\}$. Then we have that for $|x - c| < \delta$:

$$\begin{aligned} |(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

This yields that by the definition of the limit again, we have that $\lim_{x \to c}(f + g)(x) = L + M$.

$\therefore$ We have that if $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then $\lim_{x \to c}(f + g)(x) = L + M$.

A similar argument can be used to show the case for subtraction. ■

9. Let f, g be defined on A to $\mathbb{R}$ and let c be a cluster point of A.

(a) Show that if both $\lim_{x \to c} f$ and $\lim_{x \to c}(f + g)$ exist, then $\lim_{x \to c}$ exists.

Proof. Since we have that $\lim_{x \to c} f$ and $\lim_{x \to c}(f + g)$ exist, by *Theorem 4.2.4* we have that $\lim_{x \to c} g = \lim_{x \to c}((f + g) - f)$ also exists.

$\therefore\ \exists \lim_{x \to c} f \wedge \exists \lim_{x \to c}(f + g) \implies \exists \lim_{x \to c} g.$ ■

(b) If $\lim_{x \to c} f$ and $\lim_{x \to c} fg$ exist, does it follow that $\lim_{x \to c} g$ exists?

If we have that $\lim_{x \to c} f$ and $\lim_{x \to c} fg$ exist, then we have that $\lim_{x \to c} g$ doesn't necessarily have to exist. Consider the following:

Let $c = 0, f(x) = x, g(x) = \frac{1}{x}$. Then we have that $\lim_{x \to c} f(x) = \lim_{x \to 0} x = 0$, and that $\lim_{x \to c} fg(x) = \lim_{x \to 0} x \cdot \frac{1}{x} = \lim_{x \to 0} 1 = 1$. However, we have that $\lim_{x \to c} g(x) = \lim_{x \to 0} \frac{1}{x}$ does not exist.

10. Give examples of functions f and g such that f and g do not have limits at a point c, but such that both $f + g$ and fg have limits at c.

Consider the following: Let $c = 0, f(x) = \text{sgn}(x), g(x) = -\text{sgn}(x)$. By the definition of the signum function, we know that $\lim_{x \to 0} \text{sgn}$ does not exist. This yields that $\lim_{x \to 0} -\text{sgn}$ also does not exit. However, we have that $(f + g)(x) = 0$ and $(f \cdot g)(x) = -1$ for $x \neq 0$, and $(f \cdot g)(0) = 0$. This gives us that $\lim_{x \to 0}(f + g)(x) = 0$ and that $\lim_{x \to 0}(f \cdot g)(x) = -1$.

11. Determine whether the follow limits exist in $\mathbb{R}$.

(a) $\lim_{x \to 0} \sin(1/x^2) \quad (x \neq 0)$

$\lim_{x \to 0} \sin \frac{1}{x^2}$ for $x \neq 0$ does not exist.

Consider $f(x) = \sin \frac{1}{x^2}$, and $x \neq 0$. Also, let $x_n = \frac{1}{\sqrt{n\pi}}$ for $n \in \mathbb{N}$. Then we have that $\lim x_n = \frac{1}{\pi} \cdot \lim \frac{1}{\sqrt{n}} = 0$. Thus $f(x_n) = \sin \frac{1}{x_n^2} = \sin \frac{1}{\frac{1}{n\pi}} = \sin n\pi = 0$. Which yields $\lim f(x_n) = 0$.

Now, let $y_n = \frac{1}{\sqrt{(4n+1)\frac{\pi}{2}}} = \sin(4n+1)\frac{\pi}{2} = 1$. Thus $\lim f(y_n) = 1$.

Hence we have that (x_n) and (y_n) both converge to 0, however the sequences $(f(x_n))$ and $(f(y_n))$ converge to two complete separate limits.

$\therefore \lim_{x \to 0} f(x) = \lim_{x \to 0} \sin \frac{1}{x^2}$ does not exist.

(b) $\lim_{x \to 0} x \sin(1/x^2) \quad (x \neq 0)$

$\lim_{x \to 0} x \sin \frac{1}{x^2} = 0$.

Lemma 0.1. Let $A \subseteq \mathbb{R}$, let $f, g : A \to \mathbb{R}$, and let c be a cluster point of A. Suppose that $\lim_{x \to c} g(x) = 0$ and that f is bounded on some neighborhood of c. We want to show that $\lim_{x \to c} fg(x) = 0$.

Since f is bounded on some neighborhood of c, we know that $\exists\ \delta_0 > 0$ and $M > 0$ such that $|x - c| < \delta_0 \implies |f(x)| < M$.

Let $\varepsilon > 0$ be given. Since $\lim_{x \to c} g(x) = 0$, we know that $\exists\ \delta_1 > 0$ such that $|x - c| < \delta_1 \implies |g(x)| = |g(x) - 0| < \frac{\varepsilon}{M}$.

Choose $\delta := \min\{\delta_0, \delta_1\}$. Thus we now have

$$|x - c| < \delta \implies |f(x)| < M \text{ and } |g(x)| < \frac{\varepsilon}{M} \implies |f(x)g(x)| < \varepsilon$$

Thus we have that by the definition of the limit, $\lim_{x \to c} fg(x) = 0$.

□

Proof. Let $f(x) = \begin{cases} \sin \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

and let $g(x) = x$ for $x \in \mathbb{R}$.

Then we have that $|f(x)| \leq 1\ \forall\ x \in \mathbb{R}$. This gives us that f is bounded in $\mathbb{R}$. Hence $\lim_{x \to 0} g(x) = \lim_{x \to 0} x = 0$. Thus we have that by *Lemma 0.1*, we have that $\lim_{x \to 0} f(x) = 0 \implies \lim_{x \to 0} x \sin \frac{1}{x^2} = 0$. ■

13. Functions f and g are defined on R by $f(x) := x + 1$ and $g(x) := 2$ if $x \neq 1$ and $g(1) := 0$.

(a) Find $\lim_{x\to 1} g(f(x))$ and compare with the value of $g(\lim_{x\to 1} f(x))$.

We first note that $\lim_{x\to 1} f(x) = \lim_{x\to 1}(x+1) = 2$. Thus we have that $g(\lim_{x\to 1} f(x)) = g(2) = 2$.

Now, $g(f(x)) = g(x+1) = \begin{cases} 2 & x \neq 0 \\ 0 & x = 0 \end{cases}$
Thus we have that $\lim_{x\to 1} g(f(x)) = 2$. Thus we have that $\lim_{x\to 1} g(f(x)) = 2 = g(\lim_{x\to 1} f(x))$.

(b) Find $\lim_{x\to 1} f(g(x))$ and compare with the value of $f(\lim_{x\to 1} g(x))$.

$f(g(x)) = \begin{cases} f(2) & x \neq 1 \\ f(0) & x = 1 \end{cases} = \begin{cases} 3 & x \neq 1 \\ 1 & x = 1 \end{cases}$
Thus we have that $\lim_{x\to 1} f(g(x)) = 3$. Now $\lim_{x\to 1} g(x) = 2$, we have that $f(\lim_{x\to 1} g(x)) = f(2) = 3$.

$\therefore \lim_{x\to 1} f(g(x)) = f(\lim_{x\to 1} g(x))$.

8. Prove or justify if true. Provide a counterexample if false.

(a) $\lim_{x\to 3a} f(x) = 3\lim_{x\to a} f(x)$

This is a false statement. Consider the function $f(x) := \frac{1}{x}$. Then we have that if $a = 4$:

$$\lim_{x\to 3(4)} f(x) = \lim_{x\to 12} \frac{1}{x} = \frac{1}{12}$$

However, we also have the following:

$$3\lim_{x\to 4} f(x) = 3\lim_{x\to 4} \frac{1}{x} = \frac{3}{4}$$

Thus, we have that

$$\lim_{x\to 3(4)} \frac{1}{x} = \frac{1}{12} \neq \frac{3}{4} = 3\lim_{x\to 4} \frac{1}{x}$$

And thus

$$\lim_{x\to 3a} f(x) \neq 3\lim_{x\to a} f(x)$$

(b) $\lim_{x\to a} f(3x) = 3\lim_{x\to a} f(x)$

This is also a false statement. Consider $f(x) = \frac{1}{x}$ and $a = 5$. Then we have

$$\lim_{x\to 5} f(3x) = \lim_{x\to 5} \frac{1}{3x} = \frac{1}{15}$$

But

$$3 \lim_{x \to 5} f(x) = 3 \lim_{x \to 5} \frac{1}{x} = \frac{3}{5}$$

Thus we have that

$$\lim_{x \to 5} \frac{1}{3x} = \frac{1}{15} \neq \frac{3}{5} = 3 \lim_{x \to 5} \frac{1}{x}$$

Hence

$$\lim_{x \to a} f(3x) \neq 3 \lim_{x \to a} f(x)$$

(c) $\lim_{x \to 3a} f(x) = \lim_{x \to a} f(3x)$

This statement is true.

Proof. Let $y = \frac{x}{3}$. Then we have that $x = 3y$. Thus as $x \to 3a$, we get that $y \to a$. Thus $\lim_{x \to 3a} f(x) = \lim_{y \to a} f(3y) = \lim_{x \to a} f(3a)$. ■