Real Analysis Homework 6

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1. Section 3.5

2)a) Show directly from the definition that $\left(\frac{n+1}{n}\right)$ is a Cauchy sequence.

Proof. Let $a_n := (\frac{n+1}{n}) = (1 + \frac{1}{n})$ be a sequence. We want to show that $\forall\ \varepsilon > 0,\ \exists\ H(\varepsilon) \in \mathbb{N}$ s.t. $\forall\ x_n, x_m \in a_n,\ |x_n - x_m| < \varepsilon,$ for $m, n \geq H(\varepsilon)$.

Recall that if $\varepsilon > 0,\ \exists\ n_\varepsilon \in \mathbb{N}$ s.t. $0 < \frac{1}{n_\varepsilon} < \varepsilon$. So, we want to show that if we let $n_\varepsilon = H(\varepsilon)$, $\frac{1}{H(\varepsilon)} < \frac{\varepsilon}{2}$.

Let $m > n \geq H(\varepsilon)$.

So,

$$\begin{aligned}
\left|\frac{m+1}{m} - \frac{n+1}{n}\right| &= \left|\frac{1}{m} - \frac{1}{n}\right| \\
&\leq \frac{1}{m} + \frac{1}{n} && \text{by the Triangle Inequality} \\
&\leq \frac{2}{n} && \text{since } m > n \Rightarrow \frac{1}{m} < \frac{1}{n} \\
&\leq \frac{2}{H(\varepsilon)} && \text{since } n \geq H(\varepsilon) \\
&< \varepsilon && \text{since } \frac{1}{H(\varepsilon)} < \frac{\varepsilon}{2}
\end{aligned}$$

Thus, $a_n = (\frac{n+1}{n})$ is a Cauchy sequence. ■

3)b) Show directly from the definition that $\left(n + \frac{(-1)^n}{n}\right)$ is not a Cauchy sequence.

Let $x_n := n + \frac{(-1)^n}{n}$, for $n \geq 1$.

Then we have the following:

$$\begin{aligned} x_{n+1} - x_n &= (n+1) + \frac{(-1)^{n+1}}{n+1} - n - \frac{(-1)^n}{n} \\ &= 1 + \frac{(-1)^{n+1}}{n+1} - \frac{(-1)^n}{n} \\ &= 1 + (-1)^{n+1}\left(\frac{1}{n+1} + \frac{1}{n}\right) \end{aligned}$$

So,

$$\begin{aligned} x_{2m+2} - x_{2m+1} &= 1 + (-1)^{2m+1+1}\left(\frac{1}{2m+2} + \frac{1}{2m+1}\right) \\ &= 1 + \frac{1}{2m+2} + \frac{1}{2m+1} \end{aligned}$$

Thus, $|x_{2m+2} - x_{2m+1}| > 1$. Hence, if we let $\varepsilon = \frac{1}{2}$, there doesn't exist $H(\varepsilon) \in \mathbb{N}$ s.t. $|x_n - x_m| < \frac{1}{2}$, $\forall\ m, n \geq H(\varepsilon)$. This is because

$$\left|x_{2H(\varepsilon)+2} - x_{2H(\varepsilon)+1}\right| = 1 + \frac{1}{2H(\varepsilon)+2} + \frac{1}{2H(\varepsilon)+1} > 1 > \frac{1}{2}$$

Thus, $\left(n + \frac{(-1)^n}{n}\right)$ is not a Cauchy sequence.

4) Show directly from the definition that if (x_n) and (y_n) are Cauchy sequences, then $(x_n + y_n)$ and $(x_n y_n)$ are Cauchy sequences.

Proof. Recall *Lemma 3.5.4*:

Lemma. A Cauchy sequence of real numbers is bounded.

Thus, we have that (x_n) and (y_n) are bounded. By the definition of boundedness, we know that there exists $M_1, M_2 \in \mathbb{R}$ such that the following hold:

$$\exists M_1 > 0 \text{ s.t. } |x_n| \leq M_1\ \forall\ n \in \mathbb{N}$$

and

$$\exists M_2 > 0 \text{ s.t. } |y_n| \leq M_2 \forall\ n \in \mathbb{N}$$

Let $M = \max\{M_1, M_2\}$.

Since (x_n) and (y_n) are Cauchy sequences, we have the following:

$$\forall\ \varepsilon > 0,\ \exists\ H_1(\varepsilon) \in \mathbb{N} \text{ s.t. } |x_n - x_m| < \frac{\varepsilon}{2},\ \forall\ m, n \geq H_1(\varepsilon)$$

$$\forall\ \varepsilon > 0,\ \exists\ H_2(\varepsilon) \in \mathbb{N} \text{ s.t. } |y_n - y_m| < \frac{\varepsilon}{2},\ \forall\ m, n \geq H_2(\varepsilon)$$

Thus, by the Triangle Inequality, we have

$$|(x_n + y_n) - (x_m + y_m)| \leq |x_n - x_m| + |y_n - y_m|$$

Let $H(\varepsilon) = \max\{H_1(\varepsilon), H_2(\varepsilon)\}$. Then we have that $|x_n - x_m| \leq \frac{\varepsilon}{2}$ and $|y_n - y_m| \leq \frac{\varepsilon}{2}$, $\forall\ m, n \geq H(\varepsilon)$. Thus,

$$|(x_n + y_n) - (x_m - y_m)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\ \forall\ m, n \geq H(\varepsilon)$$

$\therefore$ $(x_n + y_n)$ is Cauchy.

Similarly, to show that $x_n y_n$ is Cauchy, we have

$$\begin{aligned}
|x_n y_n - x_m y_m| &= |x_n y_n - y_n x_m + x_m y_n - x_m y_m| \\
&\leq |y_n||x_n - x_m| + |x_m||y_n - y_m| \\
&\leq M(|x_n - x_m| + |y_n - y_m|) \\
&\leq M\left(\frac{\varepsilon}{2} + \frac{\varepsilon}{2}\right), && \forall\ n, m \geq H(\varepsilon) \\
&= M_\varepsilon
\end{aligned}$$

Note that we can initially replace ε by $\frac{\varepsilon}{M}$ to get $|x_n y_n - x_m y_m| \leq \varepsilon,\ \forall\ n, m \geq H(\varepsilon)$.

Thus we have that $(x_n y_n)$ is also Cauchy.

■

5) If $x_n := \sqrt{n}$, show that (x_n) satisfies $\lim |x_{n+1} - x_n| = 0$, but that it is not a Cauchy sequence.

Note that (x_n) is an unbounded sequence. By the *Cauchy Convergence Criterion*, since $\sqrt{n}$ is not bounded, (x_n) is not a Cauchy sequence. Now, we only must show that $|x_{n+1} - x_n|$ converges to 0.
So,

$$\begin{aligned}
|x_{n+1} - x_n| &= \sqrt{n+1} - \sqrt{n} \\
&= \frac{1}{\sqrt{n+1} + \sqrt{n}} \\
\Rightarrow \lim_{n\to\infty} |x_{n+1} - x_n| &= \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} \\
&= 0
\end{aligned}$$

Thus, we have that $\lim |x_{n+1} - x_n| = 0$, but (x_n) is not Cauchy.

12) If $x_1 > 0$ and $x_{n+1} := (2 + x_n)^{-1}$ for $n \geq 1$, show that (x_n) is a contractive sequence. Find the limit.

Proof. Recall the definition of a contractive sequence:

Definition. We say that a sequence (x_n) of real numbers is **contractive** if there exists a constant C, $0 < C < 1$, such that

$$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|$$

for all $n \in \mathbb{N}$. The number C is called the **constant** of the contractive sequence.

For (x_n), we have the following:

$$\begin{aligned}
|x_{n+2} - x_{n+1}| &= |(2+x_{n+1})^{-1} - (2+x_n)^{-1}| \\
&= \left| \frac{1}{2+x_{n+1}} - \frac{1}{2+x_n} \right| \\
&= \left| \frac{2 - x_n - (2+x_{n+1})}{(2+x_{n+1})(2+x_n)} \right| \\
&= \frac{|x_n - x_{n+1}|}{(2+x_{n+1})(2+x_n)} \\
&\leq \frac{|x_n - x_{n+1}|}{(2+0)(2+0)} \\
&= \frac{1}{4} \cdot |x_{n+1} - x_n|
\end{aligned}$$

Thus, if we let $C = \frac{1}{4}$, then (x_n) is a contractive sequence. ■

Now we want to find the limit of (x_n).

Recall *Theorem 3.5.8*:

Theorem. Every contractive sequence is a Cauchy sequence, and therefore is convergent.

By *Theorem 3.5.8*, we have that $\lim(x_n) = x$ exists, and since we know that $\lim(x_{n+1}) = \lim(x_n) = x$, we have the following:

$$\begin{aligned}
x_{n+1} &= (2+x_n)^{-1} \\
x_{n+1} &= \frac{1}{2+x_n} \qquad \Big| \lim \\
x &= \frac{1}{2+x} \qquad | \cdot (2+x) \\
x^2 + 2x &= 1 \\
x^2 + 2x - 1 &= 0 \\
x &= \frac{-2 \pm \sqrt{4 - 4 \cdot 1 \cdot (-1)}}{2} \\
x_a &= -1 - \sqrt{2} < 0 \\
x_b &= -1 + \sqrt{2} > 0
\end{aligned}$$

Since $x_n > 0$, $\forall\ n$, we know that $x_a < 0$ cannot be the limit, and thus we can conclude that $\lim(x_n) = -1 + \sqrt{2}$.

13) If $x_1 := 2$ and $x_{n+1} := 2 + 1/x_n$ for $n \geq 1$, show that (x_n) is a contractive sequence. What is the limit?

Lemma 0.1. We want to show that $x_n \geq 2 \; \forall \; n \in \mathbb{N}$. We prove this by method of mathematical induction.

Basis Step: Let $n = 1$. Then we have that $x_1 = 2 \geq 2$.

Inductive Step: Assume that $x_n \geq 2$ for arbitrary $n \in \mathbb{N}$.

Show: We want to now show that $x_{n+1} \geq 2, \; \forall \; n \in \mathbb{N}$. So we have the following:

$$\begin{aligned} x_{n+1} &= 2 + \frac{1}{x_n} \\ &\geq 2 + 0 \\ &= 2 \end{aligned}$$

$\therefore$ by mathematical induction, we have that $x_n \geq 2, \; \forall \; n \in \mathbb{N}$

Proof. By the definition of a contractive sequence, we have the following:

$$\begin{aligned} |x_{n+2} - x_{n+1}| &= \left| \left(2 + \frac{1}{x_{n+1}}\right) - \left(2 + \frac{1}{x_n}\right) \right| \\ &= \left| \frac{1}{x_n + 1} - \frac{1}{x_n} \right| \\ &= \left| \frac{x_n - x_{n+1}}{x_{n+1} \cdot x_n} \right| & \leq \left| \frac{x_n - x_{n+1}}{2 \cdot 2} \right| \text{by Lemma 0.1} \\ &= \frac{1}{4} |x_n - x_{n+1}| \end{aligned}$$

So, if we let $C = \frac{1}{4}$, then we have shown that by the definition of a contractive sequence, (x_n) is contractive.

∎

Now we want to find the limit of (x_n). By *Theorem 3.5.8*, we have that since (x_n) is contractive, it is also convergent. Thus, we know that $\lim(x_n) = x$ exists, and since we also know that $\lim(x_{n+1}) = \lim(x_n) = x$, we have the following:

$$\begin{aligned} x_{n+1} &= 2 + \frac{1}{x_n} & |\lim \\ x &= 2 + \frac{1}{x} & |\cdot x \\ x^2 &= 2x + 1 \\ x^2 - 2x - 1 &= 0 \\ x &= \frac{2 \pm \sqrt{4 - 4 \cdot 1 \cdot (-1)}}{2} \\ x_a &= 1 - \sqrt{2} < 2 \\ x_b &= 1 + \sqrt{2} > 2 \end{aligned}$$

By *Lemma 0.1*, we know that $x_n \geq 2, \ \forall \ n \in \mathbb{N}$, and thus we know that $x_a < 2$ can't be the limit. Thus, we can conclude that $\lim(x_n) = 1 + \sqrt{2}$.

2. Find examples of sequences of real numbers satisfying each set of properties:

(a) Cauchy, but not monotone

Consider the sequence $a_n := (1, \frac{1}{3}, \frac{1}{2}, \frac{1}{5}, \frac{1}{4}, \dots)$. Notice that $\forall \ \varepsilon > 0, \ \exists \ H(\varepsilon) \in \mathbb{N}$ s.t. $|a_n - a_m| < \varepsilon,$ for $m, n \in \mathbb{N}, \ \forall \ m, n \geq H(\varepsilon)$. Thus, this sequence satisfies the definition of a Cauchy sequence, and thus it is a Cauchy sequence. However, notice that while it is decreasing overall, it is not decreasing in a way that coincides with the definition of monotone decreasing.

That is, in order for this sequence to be considered monotone decreasing, we must have the following:

$$a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq \dots \geq a_n, \ \forall \ n \in \mathbb{N}$$

But, given the first five terms of this sequence, we have

$$1 \geq \frac{1}{3} \leq \frac{1}{2} \geq \frac{1}{5} \leq \frac{1}{4} \dots$$

Hence this sequence is Cauchy but not monotone.

(b) Monotone, but not Cauchy

Consider the sequence $a_n := (1, 4, 9, 16, 25, \dots) = n^2$. This sequence is monotone, since it is an increasing sequence, however it is not Cauchy. Consider $\varepsilon = 1$. Since (a_n) is an increasing sequence, we have that for any two $m, n \in \mathbb{N}$, $|a_n - a_m| > 1$. Thus, $\nexists \ H(\varepsilon) \in \mathbb{N}$ s.t. $\forall \ \varepsilon > 0, \ |a_n - a_m| < \varepsilon,$ for $m, n \in \mathbb{N}, \ \forall \ m, n \geq H(\varepsilon)$. Thus (a_n) is a monotone sequence but it is not a Cauchy sequence.

(c) (Section 3.5, Problem 1) Bounded, but not Cauchy

Consider the sequence $a_n := (-1, 1, -1, 1, -1, \dots) = (-1)^n$. This sequence is bounded above by 1 and is bounded below by -1, and thus this sequence is bounded. However, since $\forall \ m, n \in \mathbb{N}, \ |a_n - a_m| = |a_m - a_n| = 2$, we have that the sequence is not Cauchy. Consider $\varepsilon = 1$. Then we have that $\nexists \ H(\varepsilon) \in \mathbb{N}$ s.t. $|a_n - a_m| < \varepsilon,$ for $m, n \in \mathbb{N}, \ \forall \ m, n \geq H(\varepsilon)$. Thus, we have a sequence that is bounded, but is not a Cauchy sequence.

3. **(a)** Let $a_n = \frac{1}{2} + \frac{1}{6} + \dots + \frac{1}{n(n+1)}$. Show a_n is Cauchy.

Proof. Let $a_n = \frac{1}{2} + \frac{1}{6} + \dots + \frac{1}{n(n+1)}$. We want to show that a_n is Cauchy. That is, we want to show that $\forall \ \varepsilon > 0, \ \exists \ H(\varepsilon) \in \mathbb{N},$ s.t. $|a_m - a_n| < \varepsilon, \ \forall \ m, n \geq H(\varepsilon)$.

Let $\varepsilon > 0$ be given. Without loss of generality, let $m \geq n$. Then we have that $0 < a_m - a_n$, which yields the following:

$$
\begin{aligned}
&= \left(\frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots + \frac{1}{m(m+1)}\right) - \left(\frac{1}{2} + \cdots + \frac{1}{n(n+1)}\right) \\
&= \frac{1}{(n+1)(n+2)} + \cdots + \frac{1}{m(m+1)} \\
&= \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right) + \cdots + \left(\frac{1}{m} - \frac{1}{m+1}\right) \\
&= \frac{1}{n+1} - \frac{1}{m+1} < \frac{1}{n+1} < \frac{1}{n} = \varepsilon
\end{aligned}
$$

So, if we let $H(\varepsilon) \geq \frac{1}{\varepsilon}$, we then have that $|a_m - a_n| < \varepsilon,\ \forall\ m, n \geq H(\varepsilon)$.

$\therefore$ a_n is Cauchy. ■

(b) Let a_n satisfy $|a_n - a_{n+1}| \leq 1/3^n$. Show a_n converges.

Proof. First note that if we let $b_n := \frac{1}{3^n}$, then we have the sequence $b_n = (\frac{1}{3}, \frac{1}{9}, \frac{1}{27})$. Notice that this sequence is a Cauchy sequence. This is because if we have the following:

$$
\begin{aligned}
3^n &< 3^{2n} \\
\Rightarrow \frac{1}{3^n} &> \frac{1}{3^{2n}}
\end{aligned}
$$

Thus we have that if we let $\varepsilon = \frac{1}{3^n}$, then $|a_n - a_{n+1}| < \frac{1}{3^2 n} < \frac{1}{3^n} = \varepsilon$. So, if we let $H(\varepsilon) = \frac{\log(\frac{1}{\varepsilon})}{2\log(3)}$, then $\forall\ n \geq H(\varepsilon),\ |a_n - a_{n+1}| < \varepsilon$. Consider if we let $m = n + 1$. Then we can rewrite this as follows:

$$\forall\ \varepsilon > 0,\ \exists\ H(\varepsilon) \in \mathbb{N} \text{ s.t. } |a_n - a_m| < \varepsilon,\ \forall\ m, n \geq H(\varepsilon),\ \text{where } H(\varepsilon) = \frac{\log(\frac{1}{\varepsilon})}{2\log(3)}$$

Thus we have that any sequence a_n that satisfies this property must be a Cauchy Sequence. Thus by the *Cauchy Convergence Criteria*, we have that a_n is convergent since it is also Cauchy. Hence $\lim(a_n) = A$ exists. ■

(c) Prove that if a_n converges, then $\lim_{n\to\infty} |a_{n+1} - a_n| = 0$.

Proof. Suppose that (a_n) is a convergent sequence. Then we have that by the *Cauchy Convergence Criterion*, (a_n) is a Cauchy sequence.

Recall *Theorem 3.1.3*:

Theorem. Let $X = (x_n : n \in \mathbb{N})$ be a sequence of real numbers and let $m \in \mathbb{N}$. Then the m-tail $X_m = (x_{m+n} : n \in \mathbb{N})$ of X converges if and only if X converges. In this case, $\lim X_m = \lim X$.

Thus by *Theorem 3.1.3*, we have that if we let $m = 1$, then $a_{m+n} = a_{n+1}$ also converges since a_n converges.

Also, recall *Theorem 3.4.2*:

Theorem. If a sequence $X = (x_n)$ of real numbers converges to a real number x, then any subsequence $X' = (x_{n_k})$ of X also converges to x.

Thus we have that since the sequence (a_{n+1}) is a subsequence of (a_n), by *Theorem 3.4.2*, the subsequence (a_{n+1}) also converges to x.

Let $\lim_{n\to\infty}(a_n) = A$ and let $\lim_{n\to\infty}(a_{n+1}) = B$, for $A, B \in \mathbb{R}$.

Consider first the sequence generated by $(a_{n+1} - a_n)$.

Recall *Theorem 3.2.3*:

Theorem. **i.** Let $X = (x_n)$ and $Y = (y_n)$ be sequences of real numbers that converge to x and y, respectively, and let $c \in \mathbb{R}$. Then t he sequences $X + Y, X - Y, X \cdot Y$, and cX converge to $x + y, x - y, xy$, and cx, respectively.

ii. If $X = (x_n)$ converges to x and $Z = (z_n)$ is a sequence of nonzero real numbers that converges to z and if $z \neq 0$, then the quotient sequence X/Z converges to x/z.

Thus, by *Theorem 3.2.3*, we have

$$\lim_{n\to\infty}(a_{n+1} - a_n) = B - A$$

However, since we have that $\lim_{n\to\infty}(a_n) = x$ and by *Theorem 3.4.2*, since (a_{n+1}) is a subsequence of (a_n), we know that

$$\lim_{n\to\infty}(a_{n+1}) = \lim_{n\to\infty}(a_n) = x$$

Which yields that $A = x = B$. So, we have

$$\lim_{n\to\infty}(a_{n+1} - a_n) = B - A = A - A = B - B = x - x = 0$$

Hence,

$$\lim_{n\to\infty}|a_{n+1} - a_n| = |0| = 0$$

$\therefore$ if (a_n) converges, then $\lim_{n\to\infty}|a_{n+1} - a_n| = 0$. ∎

(d) Give an example of a sequence a_n where $\lim_{n\to\infty}|a_{n+1} - a_n| = 0$, but a_n diverges.

Consider the sequence $a_n := (\frac{1}{2}, 1, \frac{4}{3}, \frac{19}{12}, \frac{107}{60}, \frac{38}{20}, \dots) \approx (0.5,\ 1,\ 1.3333,\ 1.583333,\ 1.783333,\ 1.9,\ \dots)$. Thus we note that this sequence is monotone increasing, and also is unbounded. Thus this is a divergent sequence. However, note the resulting sequence of $|a_{n+1} - a_n|$:

$$|a_{n+1} - a_n| = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots)$$

We can note that the resulting sequence of $|a_{n+1} - a_n|$ is not only monotone decreasing, but also converges to 0. Thus we have defined a sequence (a_n) such that $\lim_{n\to\infty}(a_n) = \infty$, however $\lim_{n\to\infty}|a_{n+1} - a_n| = 0$.

(e) Let a_n satisfy $a_{n+1} = a_n^2$ for all $n \in \mathbb{N}$ where $0 < a_1 \leq 1/3$. Show a_n is contractive.

To begin, we want to show that $a_{n+1} < \frac{1}{3}$. We prove it by method of mathematical induction.

Lemma 0.2. Basis Step: Let $n = 1$. Then we have that $a_1 < \frac{1}{3}$

Inductive Step: Assume that $a_2 < \frac{1}{3}$ for arbitrary $n \in \mathbb{N}$.

Show: We want to now show that $a_{n+1} < \frac{1}{3}$, $\forall\ n \in \mathbb{N}$. So,

$$\begin{aligned} a_{n+1} = a_n^2 &< \left(\frac{1}{3}\right)^2 && \text{by definition of } a_{n+1} \\ = a_n^2 &< \frac{1}{9} < \frac{1}{3} \end{aligned}$$

Thus, by mathematical induction, we have that $a_n < \frac{1}{3}$, $\forall\ n \in \mathbb{N}$.

Now, we want to show that a_n is contractive.

Proof. By the definition of a contractive sequence, we have the following:

$$\begin{aligned} |a_{n+2} - a_{n+1}| &= a_{n+1}^2 - a_n^2| \\ &= |a_{n+1} - a_n||a_{n+1} + a_n| \\ &\leq |a_{n+1} - a_n|(|a_{n+1}| + |a_n|) \\ &\leq \frac{2}{3}|a_{n+1} - a_n| \end{aligned}$$

So, if we let $C = \frac{2}{3}$, then we have shown that by the definition of a contractive sequence, (a_n) is contractive. ■

4. Show that the following sequences are not Cauchy.

(a) $a_n = n^2$.

Consider $\varepsilon = 1$. Since (a_n) is an increasing sequence, we have that for any two $m, n \in \mathbb{N}$, $|a_n - a_m| > 1$. Thus, $\nexists\ H(\varepsilon) \in \mathbb{N}$ s.t. $\forall\ \varepsilon > 0$, $|a_n - a_m| < \varepsilon$, for $m, n \in \mathbb{N}$, $\forall\ m, n \geq H(\varepsilon)$. Thus (a_n) is not a Cauchy sequence.

(b) (Section 3.5, Problem 2b) $a_n = n + \frac{(-1)^n}{n}$.

If we let $\varepsilon = \frac{1}{2}$, there doesn't exist $H(\varepsilon) \in \mathbb{N}$ s.t. $|x_n - x_m| < \frac{1}{2}$, $\forall\, m, n \geq H(\varepsilon)$. This is because

$$\left|x_{2H(\varepsilon)+2} - x_{2H(\varepsilon)+1}\right| = 1 + \frac{1}{2H(\varepsilon)+2} + \frac{1}{2H(\varepsilon)+1} > 1 > \frac{1}{2}$$

Thus, $\left(n + \frac{(-1)^n}{n}\right)$ is not a Cauchy sequence.

5. Prove or justify, if true. Provide a counterexample, if false.

(a) If a_n is Cauchy and b_n is bounded, then $a_n \cdot b_n$ is Cauchy.

This is a false statement. Consider the sequences $(a_n) := 1$, and $b_n := (-1)^n$. We have that a_n is a Cauchy sequence and we also have that (b_n) is a bounded sequence since it is bounded below by -1 and it is bounded above by 1. Then we have that the resulting sequence of $a_n b_n$ is

$$a_n \cdot b_n = (-1, 1, -1, 1, -1, \dots)$$

Thus since this sequence oscillates between -1 and 1, it is not a Cauchy sequence since by the *Cauchy Convergence Criterion*, since this sequence doesn't converge, it isn't Cauchy.

(b) If a_n is a monotone increasing sequence such that $a_{n+1} - a_n \leq 1/n$, then a_n converges.

This is a false statement. Consider the following sequence: $a_n := (-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \dots, -\frac{1}{n})$. Thus we have that this sequence is both monotone increasing and is a harmonic sequence. Also, note for $n = 1$, we have

$$a_{n+1} - a_n = -\frac{1}{2} - (-1) = -\frac{1}{2} + 1 = \frac{1}{2} \leq \frac{1}{1} = 1$$

However, if we let $\varepsilon = \frac{1}{3}$, then we have that $|a_2 - a_1| \not< \varepsilon$, since $|a_2 - a_1| = \frac{1}{2} \not< \frac{1}{3}$. Hence this sequence is not true for all ε.

(c) The Cauchy convergence criteria holds in $\mathbb{Q}$.

This is a false statement. Consider the sequence $a_n := (1.4, 1.41, 1.414, \dots)$. We have that this sequence is indeed a Cauchy sequence, however this sequence does not converge to a value in $\mathbb{Q}$. Rather, this sequence converges to $\sqrt{2} \notin \mathbb{Q}$. Thus the Cauchy convergence criteria does not hold in $\mathbb{Q}$.