Real Analysis Homework 5

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1. For the following sequences, i) write out the first 5 terms, ii) Use the Monotone Sequence Property to show that the sequences converges.

(a) Section 3.3

2) Let $x_1 > 1$ and $x_{n+1} := 2 - 1/x_n$ for $n \in \mathbb{N}$. Show that (x_n) is bounded and monotone. Find the limit.

The first five terms of this sequence are $x_1 \geq 2, x_2 \geq \frac{3}{2}, x_3 \geq \frac{4}{3}, x_4 \geq \frac{5}{4}, x_5 \geq \frac{6}{5}, \cdots \approx x_1 \geq 2, x_2 \geq 1.5, x_3 \geq 1.3333, x_4 \geq 1.25, x_5 \geq 1.2, \ldots$. This sequence appears to be decreasing.

Recall the Monotone Sequence Property:

Theorem. Monotone Sequence Property A monotone sequence of real numbers is convergent if and only if it is bounded. Further,

A. If $X = (x_n)$ is a bounded increasing sequence, then

$$\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\}$$

B. If $Y = (y_n)$ is a bounded decreasing sequence, then

$$\lim(y_n) = \inf\{y_n : n \in \mathbb{N}\}$$

To show that this sequence converges, we must first find the possible limit points (fixed points) of this sequence. So,

$$\begin{aligned} x &= 2 - \frac{1}{x} \\ x^2 &= 2x - 1 \\ x^2 - 2x + 1 &= 0 \\ (x-1)^2 &= 0 \end{aligned}$$

Thus, $x = 1$ is a possible limit of this sequence.

Now, we will prove that (x_n) is bounded by 1, and since we hypothesized that (x_n) is decreasing, we say that (x_n) is bounded below by 1.

Proof. We want to show that the sequence (x_n) is bounded below by 1; that is, we want to show that $1 \leq x_n, \ \forall \ n \in \mathbb{N}$. We prove it by method of mathematical induction.

Basis Step: Let $n = 1$. Then

$$\begin{aligned} x_n &\geq x_{n+1}, && \text{by the definition of decreasing,} \\ x_1 &\geq x_{1+1} \\ x_1 &\geq x_2 \end{aligned}$$

Since $x_1 > 1 \Rightarrow \frac{1}{x_1} < 1$, we have

$$x_2 = 2 - \frac{1}{x_1} > 1$$

$$\Rightarrow 1 < x_2 < 2.$$

Since $x_1 > 1$ and because $1 < x_2 < 2$, we have that $x_1 \geq x_2$.

Inductive Step: Assume $1 < x_n < 2, \ \forall \ n \in \mathbb{N}$.

Show: Now we want to show that $x_n \leq x_{n+1}$.
So,

$$\begin{gathered} 1 < x_n < 2 \\ 1 > \frac{1}{x_n} > \frac{1}{2} \\ -1 < -\frac{1}{x_n} < -\frac{1}{2} \\ 1 < 2 - \frac{1}{x_n} < 2 - \frac{1}{2} < 2 \\ 1 < x_{n+1} < 2 \end{gathered}$$

Thus we have that (x_n) is bounded between 1 and 2. ■

Now we need to show that (x_n) is monotone decreasing; that is, we must show that $x_1 \geq x_2 \geq \cdots \geq x_n$.

Proof. We want to show that $x_1 \geq x_2 \geq \cdots \geq x_n, \ \forall \ n \in \mathbb{N}$. We prove it by method of mathematical induction.

Basis Step: Let $n = 1$. Then since $x_1 > 1$ is given, we have that $\frac{1}{x_1} < 1$. This yields $x_2 = 2 - \frac{1}{x_1} > 1$, as was determined for the boundedness proof, and thus we have that $1 < x_2 < 2$. This means that $1 > \frac{1}{x_2} > \frac{1}{2}$, and since $\frac{1}{2} \leq \frac{1}{x_n}$, we have $x_2 \geq x_1$.

Inductive Step: Assume $x_n \geq x_{n+1} \ \forall \ n \in \mathbb{N}$.

Show: We now want to show that $x_{n+2} \leq x_{n+1}$.
So,

$$x_{n+2} = 2 - \frac{1}{x_{x+1}}$$

Recall the inductive hypothesis, in that $x_n \geq x_{n+1} \Rightarrow \frac{1}{x_n} \leq \frac{1}{x_{n+1}}$. Thus,

$$-\frac{1}{x_n} \geq -\frac{1}{x_{n+1}}$$

$$\Rightarrow 2 - \frac{1}{x_n} \leq 2 - \frac{1}{x_{n+1}}$$

$$x_{n+1} \leq x_{n+2}$$

$\therefore$ we have that $x_1 \geq x_2 \geq \cdots \geq x_n, \ \forall\, n \in \mathbb{N}$. ■

Thus (x_n) is monotone decreasing.

By the *Monotone Sequence Property*, since we have shown that (x_n) is both bounded (and thus converges), and that (x_n) is monotone decreasing, we have that

$$\begin{aligned} \lim(x_n) &= \inf\{x_n : n \in \mathbb{N}\} \\ &= \inf(1, 2) \\ &= 1 \end{aligned}$$

Hence the sequence converges to the previously found possible limit of 1.

3) Let $x_1 > 1$ and $x_{n+1} := 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$. Show that (x_n) is decreasing and bounded below by 2. Find the limit.

The first 5 terms of this sequence are $x_1 \geq 2, x_2 \geq 2, x_3 \geq 2, x_4 \geq 2, x_5 \geq 2, \ldots$. Notice the following, however:

$$\begin{aligned} x_{n+1} \leq x_n &\iff 1 + \sqrt{x_n - 1} \leq x_n \\ &\iff \sqrt{x_n - 1} \leq x_n - 1 \end{aligned}$$

which we know is always true since the square root function is a decreasing function.

Now we must find the possible limit points (fixed points) of this sequence. So,

$$\begin{aligned}
x &= 1 + \sqrt{x-1} \\
x - 1 &= \sqrt{x-1} \\
x - 1 &= (x-1)^2 \\
x - 1 &= x^2 - 2x + 1 \\
(x-1) - (x^2 - 2x + 1) &= 0 \\
-x^2 + 3x - 2 &= 0 \\
-(x^2 - 3x + 2) &= 0 \\
-(x-1)(x-2) &= 0 \\
(x-1)(x-2) &= 0
\end{aligned}$$

Thus $x = 1$, or $x = 2$. These are the possible limits of (x_n). Since we hypothesized that (x_n) is decreasing, then we say that (x_n) is bounded below by 2, since we are given that $x_1 > 1$.

Now we will prove that (x_n) is bounded below by 2.

Proof. We want to show that (x_n) is bounded below by 1; that is, we want to show that $1 \leq x_n$, $\forall\ n \in \mathbb{N}$. We prove it by method of mathematical induction.

Basis Step: Let $n = 1$. Then we are given that $x_1 \geq 2$.

Inductive Step: Assume that $x_n \geq 2$, $\forall\ n \in \mathbb{N}$.

Show: We now want to show that $x_{n+1} \geq 2$, $\forall\ n \in \mathbb{N}$.

So,

$$\begin{aligned}
x_{n+1} &= 1 + \sqrt{x_n - 1} \\
&\geq 1 + \sqrt{2-1} \\
&= 1 + 1 \\
&= 2
\end{aligned}$$

Thus, $x_n \geq 2$, $\forall\ n \in \mathbb{N}$. By the definition of boundedness, we have that (x_n) is bounded below by 2. ■

Since we have also shown earlier that (x_n) is monotone decreasing, we have that by the monotone sequence property, since (x_n) is bounded, (x_n) converges, and since (x_n) is monotone decreasing, we have:

$$\begin{aligned}
\lim(x_n) &= \inf\{x_n : n \in \mathbb{N}\} \\
&= 2
\end{aligned}$$

7) Let $x_1 := a > 0$ and $x_{n+1} := x_n + 1/x_n$ for $n \in \mathbb{N}$. Determine whether (x_n) converges or diverges.

The first 5 terms of this sequence are $x_1 \geq 1, x_2 \geq 2, x_3 \geq \frac{5}{2}, x_4 \geq \frac{29}{10}, x_5 \geq \frac{941}{290}, \cdots \approx x_1 \geq 1, x_2 \geq 2, x_3 \geq 2.5, x_4 \geq 2.9, x_5 \geq 3.244828, \ldots$. This sequence appears to be increasing. We show this to be true as follows:

$$\begin{aligned} x_{n+1} \geq x_n &\iff x_n + \frac{1}{x_n} \geq x_n \\ &\iff x_n^2 + 1 \geq x_n^2 \\ &\iff 1 \geq 0 \end{aligned}$$

which is true. However, notice that one of the terms of the sequence is x_n. We know that x_n is an unbounded sequence. Thus, we can infer that (x_n) is unbounded above. We show this as follows:

$$\begin{aligned} x_{n+1}^2 &= \left(x_n + \frac{1}{x_n}\right)^2 \\ &= x_n^2 + 2 + \frac{1}{x_n^2} \\ &> x_n^2 + 2 \end{aligned}$$

Since:

$$x_{n+1}^2 > x_n^2 + 2 > x_{n-1}^2 + 4 > \cdots > x_1^2 + 2 \cdot n = a^2 + 2 \cdot n$$
$$\Downarrow$$
$$x_n > \sqrt{a^2 + 2 \cdot (n-1)}$$

Since the right hand side of this inequality is unbounded, the left hand side is also unbounded.

Thus we have that this sequence (x_n) is unbounded above.

Since this sequence is increasing and unbounded above, we have that the sequence is divergent.

8) Let (a_n) be an increasing sequence, (b_n) be a decreasing sequence, and assume that $a_n \leq b_n$ for all $n \in \mathbb{N}$. Show that $\lim(a_n) \leq \lim(b_n)$, and thereby deduce the Nested Intervals Property 2.5.2 from the Monotone Convergence Theorem 3.3.2.

Since (a_n) is an increasing sequence, we know that $(a_1 \leq a_2 \leq \cdots \leq a_n)$, and since (b_n) is a decreasing sequence, we know that $(b_1 \geq b_2 \geq \cdots \geq b_n)$. Also, since we have that $a_n \leq b_n,\ \forall\ n \in \mathbb{N}$, we know that (a_n) is bounded above by (b_1). Thus, by the *Monotone Convergence Theorem*, we know that

$$\lim(a_n) = \sup\{a_n : n \in \mathbb{N}\}$$

Also, since (b_n) is a decreasing sequence such that it is bounded below by (a_1), by the *Monotone Convergence Theorem*, we have

$$\lim(b_n) = \inf\{b_n : n \in \mathbb{N}\}$$

Recall Theorem 3.2.5:

Theorem. If $X = (x_n)$ and $Y = (y_n)$ are convergent sequences of real numbers and if $x_n \leq y_n \ \forall \ n \in \mathbb{N}$, then $\lim(x_n) \leq \lim(y_n)$.

Also, recall the *Nested Intervals Property*:

Theorem. If $I_n = [a_n, b_n], \ n \in \mathbb{N}$, is a nested sequence of closed bounded intervals, then there exists a number $\xi \in \mathbb{R}$ s.t. $\xi \in I_n \ \forall \ n \in \mathbb{N}$.

Note that we have a nested sequence of closed, bounded intervals: $[a_n, b_n], \ n \in \mathbb{N}$. Since we showed that $\lim(a_n) \leq \lim(b_n)$, (and we are given that (a_n) is increasing and (b_n) is decreasing), we know that there exists ξ such that

$$\lim(a_n) \leq \xi \leq \lim(b_n)$$

which means that $\xi \in [a_n, b_n], \ \forall \ n \in \mathbb{N}$.

(b) $a_1 = 1, \ a_{n+1} = \frac{a_n^2+5}{2a_n}$

The first 5 terms of this sequence are $1, 3, \frac{7}{3}, \frac{47}{21}, \frac{2207}{987}, \cdots \approx 1, 3, 2.3333, 2.2381, 2.2361, \ldots$. This is a decreasing sequence.

First, we must find the possible limits (fixed points) of the sequence. So,

$$\begin{aligned} a &= \frac{a^2+5}{2a} \\ 2a^2 &= a^2 + 5 \\ a^2 &= 5 \\ a &= \pm\sqrt{5} \end{aligned}$$

Since we're given that $a_1 = 1$, we know that the most likely lower bound will be $\sqrt{5}$.

Now we want to show that (a_n) is bounded below by $\sqrt{5}$.

Proof. We want to show that $a_n \geq \sqrt{5}, \ \forall \ n \in \mathbb{N}$. We prove it by method of mathematical induction.

Basis Step: Since $1 \geq \sqrt{5}$, we have that $a_1 \geq \sqrt{5}$

Inductive Step: Assume that $a_n \geq \sqrt{5} \ \forall \ n \in \mathbb{N}$.

Show: We want to show that $a_{n+1} \geq \sqrt{5} \ \forall \ n \in \mathbb{N}$. So,

$$a_{n+1} = \frac{a_n^2+5}{2a_n}$$

$$
\begin{aligned}
(a_n - \sqrt{5})^2 &\geq 0 \\
a_n^2 - 2\sqrt{5}a_n + 5 &\geq 0 \\
a_n^2 + 5 &\geq 2\sqrt{5}a_n \\
&\Downarrow \\
\frac{a_n^2 + 5}{2a_n} &\geq \frac{2\sqrt{5}a_n}{2a_n} \\
\frac{a_n^2 + 5}{2a_n} &\geq \sqrt{5} \\
a_{n+1} &\geq \sqrt{5}
\end{aligned}
$$

Thus we have that (a_n) is bounded below by $\sqrt{5}$. ■

Now we must show that (a_n) is monotone decreasing.

Proof. We want to show that (a_n) is monotone decreasing; that is, we want to show that $(a_2 \geq a_3 \geq \cdots \geq a_n)$, $\forall\ n \geq 2$. We prove it by method of mathematical induction.

Basis Step: Since $3 \geq \frac{7}{3}$, we have that $a_2 \geq a_3$.

Inductive Step: Assume that $a_n \geq a_{n+1}$, $\forall\ n \geq 2$.

Show: We want to show that $a_{n+2} \leq a_{n+1}$, $\forall\ n \geq 2$.
So,

$$a_{n+2} = \frac{a_{n+1}^2 + 5}{2a_{n+1}} \leq \frac{a_n^2 + 5}{2a_n}$$

Since we have:

$$
\begin{aligned}
&a_{n+1} \geq \sqrt{5}, && \text{by the previous proof of boundedness} \\
&a_{n+1}^2 \geq 5
\end{aligned}
$$

We can equivalently write the inequality as

$$\frac{a_{n+1}^2 + 5}{2a_{n+1}} \leq \frac{a_{n+1}^2 + a_{n+1}^2}{2a_{n+1}} = a_{n+1}$$

Thus we have that (a_n) is monotone decreasing. ■

Since (a_n) is both monotone decreasing and bounded, we have

$$
\begin{aligned}
\lim(a_n) &= \inf\{a_n : n \in \mathbb{N}\} \\
&= \sqrt{5}
\end{aligned}
$$

(c) $a_1 = 5,\ a_{n+1} = \sqrt{4 + a_n}$

The first 5 terms of this sequence are 5, 3, $\sqrt{7}$, $\frac{\sqrt{14}}{2} + \frac{\sqrt{2}}{2}$, $\frac{\sqrt{2\cdot(\sqrt{14}+\sqrt{2}+8)}}{2}$, $\ldots$, $\approx$ 5, 3, 2.64575131106, 2.57793547457, 2.5647486182, $\ldots$. This sequence is decreasing.

First, we must find the possible limits (fixed points) of the sequence. So,

$$\begin{aligned}
a &= \sqrt{4+a} \\
\sqrt{4+a} &= a \\
4 + a &= a^2 \\
-a^2 + a + 4 &= 0 \\
a^2 - a - 4 &= 0 \\
a^2 - a &= 4 \\
a^2 - a + \frac{1}{4} &= 4 + \frac{1}{4} \\
a^2 - a + \frac{1}{4} &= \frac{17}{4} \\
(a - \frac{1}{2})^2 &= \frac{17}{4} \\
a - \frac{1}{2} &= \pm\frac{\sqrt{17}}{2}
\end{aligned}$$

So we have that $a = \frac{1}{2} + \frac{\sqrt{17}}{2}$, or $a = \frac{1}{2} - \frac{\sqrt{17}}{2}$. We must now check these solutions for correctness; so,

$$\begin{aligned}
a \Rightarrow \frac{1}{2} - \frac{\sqrt{17}}{2} &= \frac{1}{2}\left(1 - \sqrt{17}\right) \\
&\approx -1.56155
\end{aligned}$$

$$\begin{aligned}
\sqrt{a+4} &= \sqrt{\left(\frac{1}{2} - \frac{\sqrt{17}}{2}\right) + 4} \\
&= \frac{\sqrt{9 - \sqrt{17}}}{\sqrt{2}} \\
&\approx 1.56155
\end{aligned}$$

Thus, this solution is incorrect. Now we must validate that $a = \frac{1}{2} + \frac{\sqrt{17}}{2}$ is correct.

So,

$$a \Rightarrow \frac{1}{2} + \frac{\sqrt{17}}{2} = \frac{1}{2}\left(1 + \sqrt{17}\right)$$
$$\approx 2.56155$$

$$\sqrt{a+4} = \sqrt{\left(\frac{\sqrt{17}}{2} + \frac{1}{2}\right) + 4}$$
$$= \frac{\sqrt{9 + \sqrt{17}}}{\sqrt{2}}$$
$$\approx 2.56155$$

Thus $a = \frac{1}{2} + \frac{\sqrt{17}}{2}$ is a correct solution.

Now we want to show that (a_n) is bounded below by $\frac{1}{2} + \sqrt{17}$.

Proof. We want to show that $a_n \geq \frac{1}{2} + \frac{\sqrt{17}}{2}$, $\forall\ n \in \mathbb{N}$, by the definition of a lower bound. We prove this by method of mathematical induction.

Basis Step: Since $5 \geq \frac{1}{2} + \frac{\sqrt{17}}{2}$, we have that $a_1 \geq \frac{1}{2} + \frac{\sqrt{17}}{2}$.

Inductive Step: Assume $a_n \geq \frac{1}{2} + \frac{\sqrt{17}}{2}$, $\forall\ n \in \mathbb{N}$.

Show: We now want to show that $a_{n+1} \geq \frac{1}{2} + \frac{\sqrt{17}}{2}$ $\forall\ n \in \mathbb{N}$. So,

$$\begin{aligned}
a_{n+1} &= \sqrt{4 + a_n}, && \text{by the definition of the sequence} \\
&\geq \sqrt{4 + \left(\frac{1}{2} + \frac{\sqrt{17}}{2}\right)}, && \text{by the inductive hypothesis} \\
&\geq \sqrt{\frac{8}{2} + \frac{1}{2} + \frac{\sqrt{17}}{2}} \\
&\geq \sqrt{\frac{9 + \sqrt{17}}{2}} \\
&\geq \sqrt{\frac{1}{2}\left(9 + \sqrt{17}\right)} \\
&\geq \sqrt{\frac{1}{4} + \frac{\sqrt{17}}{2} + \frac{17}{4}}, && \text{by expressing } \frac{9 + \sqrt{17}}{2} \text{ as a square} \\
&\geq \sqrt{\frac{1 + 2\sqrt{17} + 17}{4}} \\
&\geq \sqrt{\frac{1 + 2\sqrt{17} + (\sqrt{17})^2}{4}} \\
&\geq \sqrt{\frac{(\sqrt{17} + 1)^2}{4}} \\
&\geq \sqrt{\frac{1}{4}\left(1 + \sqrt{17}\right)^2} \\
&\geq \frac{\sqrt{(1 + \sqrt{17})^2}}{\sqrt{4}} \\
&\geq \frac{\sqrt{17} + 1}{2} \\
&\geq \frac{1}{2} + \frac{\sqrt{17}}{2}
\end{aligned}$$

Thus we have that $a_{n+1} \geq \frac{1}{2} + \frac{\sqrt{17}}{2}$ $\forall\ n \in \mathbb{N}$. ■

Now, we want to show that (a_n) is monotone decreasing; that is, we want to show that $(a_1 \geq a_2 \geq \cdots \geq a_n)$.

Proof. We want to show that $(a_1 \geq a_2 \geq \cdots \geq a_n)$, $\forall\ n \in \mathbb{N}$. We prove this by method of mathematical induction.

Basis Step: Since $5 \geq 3$, we have that $a_1 \geq a_2$.

Inductive Step: Assume $a_n \geq a_{n+1} \; \forall \; n \in \mathbb{N}$.

Show: We want to show that $a_{n+1} \geq a_{n+2} \; \forall \; n \in \mathbb{N}$. So,

$$\begin{aligned} a_{n+2} &= \sqrt{4 + a_{n+1}} && \text{by the definition of the sequence} \\ &\leq \sqrt{4 + a_n} && \text{by the inductive hypothesis} \\ &= a_{n+1} \end{aligned}$$

Thus we have that $a_{n+1} \geq a_{n+2} \; \forall \; n \in \mathbb{N}$. ■

Since (a_n) is both bounded and monotone decreasing, by the *Monotone Convergence Theorem*, we have that (a_n) converges. Also by the *Monotone Sequence Property*, we have that (a_n) converges to the following:

$$\begin{aligned} \lim(a_n) &= \inf\{a_n : n \in \mathbb{N}\} \\ &= \frac{1}{2} + \frac{\sqrt{17}}{2} \approx 2.56155281281 \end{aligned}$$

2. **(a)** Show $a_n = \frac{3 \cdot 5 \cdot 7 \cdot \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots (2n)}$ converges to A where $0 \leq A < 1/2$.

First, we note the first few terms of this sequence: $\frac{3}{2}, \frac{15}{8}, \frac{105}{48}, \ldots$. Now, since we have that

$$0 < \frac{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots 2n} < \frac{1}{2} \cdot 1 \cdot 1 \cdot \cdots \cdot 1 = \frac{1}{2}$$

We have that a_n is bounded. Also, we note that a_n is strictly decreasing since $\frac{a_{n+1}}{a_n} = \frac{2n-1}{2n} < 1$ (i.e. $a_{n+1} < a_n$) Thus by the *Monotone Sequence Property*, we have that a_n converges to A where $0 \leq A < \frac{1}{2}$.

(b) Show $b_n = \frac{2 \cdot 4 \cdot 6 \ldots (2n)}{3 \cdot 5 \cdot 7 \ldots (2n+1)}$ converges to B where $0 \leq B < 2/3$.

To begin, note that the first few terms of this sequence are $\frac{2}{3}, \frac{8}{15}, \frac{48}{105}, \ldots$. Now, since we have that

$$0 < \frac{2 \cdot 4 \cdot 6 \cdot \ldots (2n)}{3 \cdot 5 \cdot 7 \cdot \ldots (2n-1)} < \frac{2}{3} \cdot 1 \cdot 1 \cdot \cdots \cdot 1 = \frac{2}{3}$$

Thus we have that b_n is bounded, and that it's strictly decreasing since $\frac{b_{n+1}}{b_n} = \frac{2n}{2n+1} < 1$. That is, $b_{n+1} < b_n$. Hence by the *Monotone Sequence Property*, we have that b_n converges to B where $0 \leq B < \frac{2}{3}$.

3. **Section 3.4**

1) Give an example of an unbounded sequence that has a convergent subsequence.

Consider the sequence $(a_n) = (1, 1, 2, 1, 3, 1, 4, 1, \ldots)$ Clearly, this sequence is bounded below by 1 but is unbounded above, and thus this sequence is unbounded. However, consider the subsequence (a_{2n-1}). Then the resulting sequence is $(a_{2n-1}) = (1, 1, 1, 1, 1, \ldots)$. Since this is a constant sequence, we have that (a_{2n-1}) converges to 1. And hence we have an unbounded sequence that has a convergent subsequence.

3) Let (f_n) be the Fibonacci sequence of Example 3.1.2(d), and let $x_n := f_{n+1}/f_n$. Given that $\lim(x_n) = L$ exists, determine the value of L.

We can rewrite (x_n) as follows:

$$\begin{aligned}
x_n &= \frac{f_{n+1}}{f_n} \\
&= \frac{f_n + f_{n-1}}{f_n} \\
&= 1 + \frac{f_{n-1}}{f_n} \\
&= 1 + \frac{\frac{1}{f_n}}{f_{n-1}} \\
&= 1 + \frac{1}{x_{n-1}}
\end{aligned}$$

Since we're given that $L = \lim(x_n)$ exists and since we just showed that it's equal to $\lim(x_{n-1})$, we get the following:

$$\begin{aligned}
x_n &= 1 + \frac{1}{x_{n-1}} \qquad \Big| \lim \\
\lim(x_n) &= 1 + \frac{1}{\lim(x_{n-1})} \\
L &= 1 + \frac{1}{L} \qquad \Big| \cdot L \\
L^2 &= L + 1 \\
L^2 - L - 1 &= 0 \\
L_{1,2} &= \frac{1 \pm \sqrt{1 - 4 \cdot 1 \cdot (-1)}}{2} \\
L_1 &= \frac{1 - \sqrt{5}}{2} < 0 \\
L_2 &= \frac{1 + \sqrt{5}}{2} > 0
\end{aligned}$$

Now, since $f_n > 0 \Rightarrow x_n > 0 \Rightarrow L > 0$, we can infer that the proper limit is

$$L = \frac{1 + \sqrt{5}}{2}$$

4a) Show that the sequence $(1 - (-1)^n + 1/n)$ converges.

Let $(x_n) := (1 - (-1)^n + 1/n)$. Let $(z_n) = (x_{2n})$, and $(w_n) = (x_{2n-1})$ be subsequence of (x_n). Then (z_n) is the subsequence of all terms of (x_n) such that n is even, and (w_n) is the subsequence of all terms of (x_n) such that n is odd.

These subsequences yield the following:

$$z_n = x_{2n} = 1 - (-1)^{2n} + \frac{1}{2n} = 1 - 1 + \frac{1}{2n} = \frac{1}{2n}$$

$$w_n = x_{2n-1} = 1 - (-1)^{2n-1} + \frac{1}{2n-1} = 1 + 1 + \frac{1}{2n-1} = 2 + \frac{1}{2n-1}$$

Now, if we take the limit of each sequence as $n \to \infty$ yields

$$\lim_{n\to\infty}(z_n) = 0 \neq 2 = \lim_{n\to\infty}(w_n)$$

Recall Theorem 3.4.5 *Divergence Criteria*:

Theorem. If a sequence $X = (x_n)$ of real numbers has either of the following properties, then X is divergent.

i. X has two convergent subsequences $X' = (x_{n_k})$ and $X'' = (x_{r_k})$ whose limits are not equal.

ii. X is unbounded

Thus by the *Divergence Criteria*, we have that since (z_n) and (w_n) satisfy the first property of the *Divergence Criteria*, we can conclude that the sequence (x_n) is divergent.

16) Give an example to show that Theorem 3.4.9 fails if the hypothesis that X is a bounded sequences is dropped.

Recall *Theorem 3.4.9*:

Theorem. Let $X = (x_n)$ be a bounded sequence of real numbers and let $x \in \mathbb{R}$ have the property that every convergent subsequence of X converges to x. Then the sequence X converges to x.

Consider the sequence $a_n = (0, -1, 0, -2, 0, -3, 0, -4, \ldots)$. Note that the subsequence $a_{2n-1} = (0, 0, 0, 0, 0, \ldots)$. Thus, $\lim(a_{2n-1} = 0)$. However, since (a_n) is not bounded, we have also that $\lim(a_n) \neq 0$, since (a_n) is divergent. It is only bounded above by 0, but it is not bounded below.

18) Show that if (x_n) is a bounded sequence, then (x_n) converges if and only if $\limsup(x_n) = \liminf(x_n)$.

Proof. Let (x_n) be a bounded sequence. We want to show that (x_n) converges if and only if $\limsup(x_n) = \liminf(x_n)$.

$(\Rightarrow)$ Suppose that (x_n) is a bounded sequence, and suppose that (x_n) converges. We want to show that $\limsup(x_n) = \liminf(x_n)$.

Recall *Theorem 3.4.2*:

Theorem. If a sequence $X = (x_n)$ of real numbers converges to a real number x, then any subsequence $X' = (x_{n_k})$ of X also converges to x.

By *Theorem 3.4.2*, we have that (x_n) has one and only one limit, x. Thus we have that $\limsup(x_n) = \liminf(x_n)$.

($\Leftarrow$) Conversely, suppose that $\limsup(a_n) = \liminf(a_n)$. Recall the definitions of the supremum and infimum for some nonempty subset S of the real numbers:

$$\sup(S) = u \Leftrightarrow \; i) \;\; \forall\, s \in S, \; u \geq s, \; \wedge \;\; ii) \; \forall\, \varepsilon > 0, \; \exists s_\varepsilon \in S \text{ s.t. } u - \varepsilon < s_\varepsilon$$

and

$$\inf(S) = w \Leftrightarrow \;\; i) \; \forall\, s \in S, \; w \leq s, \; \wedge \;\; ii) \; \forall\, \varepsilon > 0, \; \exists\, s_\varepsilon \in S \text{ s.t. } s_\varepsilon < w + \varepsilon$$

Also, recall the definition of the limit of a sequence for some sequence (a_n):

$$\lim(a_n) = A \implies \;\; \forall\, \varepsilon > 0, \; \exists\, K(\varepsilon) \in \mathbb{N} \text{ s.t. } |a_n - A| < \varepsilon, \; \forall\, n \geq K(\varepsilon)$$

So, if we let $\limsup x_n = a$, then we know that there exists a natural number $K(\varepsilon_1)$ such that $x_n < a + \varepsilon, \; \forall\, n \geq K(\varepsilon_1)$. Also, for $\liminf(x_n) = a$ yields that $\exists\, K(\varepsilon_2) \in \mathbb{N}$ s.t. $x_n > a - \varepsilon \; \forall\, n \geq K(\varepsilon_2)$. Now, let $K(\varepsilon) = \max\{K(\varepsilon_1), K(\varepsilon_2)\}$. Then $a - \varepsilon < x_n < a + \varepsilon \; \forall\, n \geq K(\varepsilon) \; \Rightarrow \; |x_n - a| < \varepsilon \; \forall\, n \geq K(\varepsilon)$. Thus, by definition we have that $\lim x_n = a$. $\therefore$ we have that x_n is convergent. ■

19) Show that if (x_n) and (y_n) are bounded sequences, then

$$\limsup(x_n + y_n) \leq \limsup(x_n) + \limsup(y_n).$$

Give an example in which the two sides are not equal.

Proof. We first note that since (x_n) and (y_n) are bounded sequences of real numbers, we have that $\{x_n + y_n\}$ is also bounded. Let $\limsup x_n = a_1, \limsup(y_n) = a_2$, and $\limsup(x_n + y_n) = p$. Let $\varepsilon > 0$. Since $\limsup(x_n) = a_1$, we know that there exists $K(\varepsilon_1) \in \mathbb{N}$ s.t. $x_n < a_1 + \frac{\varepsilon}{2}, \; \forall\, n \geq K(\varepsilon_1)$. We also know that since $\limsup(y_n) = a_2, \; \exists\, K(\varepsilon_2) \in \mathbb{N}$ s.t. $y_n < a_2 + \frac{\varepsilon}{2}, \; \forall\, n \geq K(\varepsilon_2)$.
Let $K(\varepsilon) = \max\{K(\varepsilon_1), K(\varepsilon_2)\}$. Then we have that $x_n < a_1 + \frac{\varepsilon}{2}, \; y_n < a_2 + \frac{\varepsilon}{2} \;\; \forall\, n \geq K(\varepsilon) \; \Rightarrow \; x_n + y_n < a_1 + a_2 + \varepsilon, \; \forall\, n \geq K(\varepsilon)$. Hence no subsequential limit of $(x_n + y_n)$ can be greater than $a_1 + a_2 + \varepsilon$.

Thus $p \leq a_1 + a_2 + \varepsilon$. Now since $\varepsilon > 0$ is arbitrary, we have that $p \leq a_1 + a_2$.
So $\limsup(x_n + y_n) \leq \limsup(x_n) + \limsup(y_n)$. ■

Example: Let $x_n = \sin(\frac{n\pi}{2})$, and let $y_n = \cos(\frac{n\pi}{2})$, for $n \in \mathbb{N}$. Then we have that (x_n) is a sequence of 0, 1, and -1. Additionally, $x_{4n+1} = \sin(\frac{(4n+1)\pi}{2}) = 1, \; \forall\, n \in \mathbb{N}$. Thus $\lim(x_{4n+1}) = 1$. Therefore we have that 1 is a subsequential limit of (x_n). Also, we have that 1 is the greatest subsequential limit since $1 > 0$ and $1 > -1$. So, $\limsup(x_n) = 1$ and by similar logic we have that $\limsup(y_n) = 1$, and $\limsup(x_n + y_n) = 1$ because $(x_n + y_n)$ and (y_n) are also sequence of $-1, 0, 1$, and $\lim(y_{4n}) = \lim \cos 2n\pi = 1$, $\lim(x_{4n+1} + y_{4n+1}) = 1$. Then $\limsup(x_n + y_n) = 1 < 2 = \limsup(x_n) + \limsup(y_n)$.

4. (a) Show that $x_n = e^{\sin(5n)}$ has a convergent subsequence.

Let $y_n = x_{\frac{n\pi}{10n}+\frac{2\pi n}{5}}$.

Then we have that the first 5 terms of is subsequence of (x_n) are $e^{\sin(\frac{5\pi}{2})}$, $e^{\sin(\frac{9\pi}{2})}$, $e^{\sin(\frac{13\pi}{2})}$, $e^{\sin(\frac{17\pi}{2})}$, $e^{\sin(\frac{21\pi}{2})}$, $\cdots = e^1, e^1, e^1, e^1, e^1, \cdots = e, e, e, e, e, \ldots$. Thus we have that the subsequence (y_n) converges to e.

(b) Give an example of a bounded sequence with three subsequences converging to three different numbers.

Let $a_n = (n \mod 3 + 1)$ be a bounded sequence. We have that (a_n) is bounded above by $\frac{10}{3}$ and is bounded below by 2. The first five terms of this sequence are $\frac{5}{2}$, $\frac{10}{3}$, 2, $\frac{5}{2}$, $\frac{10}{3}$, ... Now, let $x_n = a_{3n+1}$, $y_n = a_{3n+2}$, and $z_n = a_{3n+3}$. Thus for each of the sequences, we have the following:

$$x_n = (\frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \ldots)$$

$$y_n = (\frac{10}{3}, \frac{10}{3}, \frac{10}{3}, \frac{10}{3}, \frac{10}{3}, \ldots)$$

$$z_n = (2, 2, 2, 2, 2, \ldots)$$

By this, we can conclude that (x_n) converges to $\frac{5}{2}$, (y_n) converges to $\frac{10}{2}$, and that (z_n) converges to 2. Hence we have that there exist three different subsequences that converge to three different numbers.

(c) Give an example of a sequence x_n with $\limsup x_n = 5$ and $\liminf x_n = -3$.

Consider the sequence $a_n = (-3, 5, -3, 5, -3, 5, -3, 5, \ldots)$. For the subsequences $b_n = a_{2n-1} = (-3, -3, -3, -3, -3, \ldots)$, and $c_n = a_{2n} = (5, 5, 5, 5, 5, \ldots)$. Then we have the following:

$$\begin{array}{rrrr} \inf(a_n) = -3 & \sup(a_n) = 5 & \liminf(a_n) = -3 & \limsup(a_n) = 5 \\ \inf(b_n) = -3 & \sup(b_n) = -3 & \liminf(b_n) = -3 & \limsup(b_n) = -3 \\ \inf(c_n) = 5 & \sup(c_n) = 5 & \liminf(c_n) = 5 & \limsup(c_n) = 5 \end{array}$$

(d) Let $\limsup x_n = 2$. True or False: if n is sufficiently large, then $x_n > 1.99$.

This statement is true. Let (x_{n_k}) be a subsequence of (x_n) such that $\lim(x_{n_k}) = 2$. Then we have that if (x_{n_k}) decreases to 2, then any element of (x_{n_k}) is going to be greater than 1.99. If (x_{n_k}) is a constant sequence, then the same is also true. Lastly, if (x_{n_k}) is increasing to 2, then we have that elements of (x_{n_k}) must have an arbitrary distance ε between the elements themselves and the limit of 2. Thus, we have that $\forall\ \varepsilon > 0,\ \exists\ K(\varepsilon) \in \mathbb{N}$ s.t. $|x_{n_k} - 2| < \varepsilon,\ \forall\ n_k \geq K(\varepsilon)$, by the definition of the limit of a sequence.

(e) Compute the infimum, supremum, limit infimum, and limit supremum for $a_n = 3 - (-1)^n - (-1)^n/n$.

$$\inf(a_n) = 1.5$$

$$\sup(a_n) = 5$$

$$\liminf(a_n) = 2$$

$$\limsup(a_n) = 4$$

5. Prove or justify, if true. Provide a counterexample, if false.

(a) If a_n and b_n are strictly increasing, then $a_n + b_n$ is strictly increasing.

Proof. Since (a_n) and (b_n) are strictly increasing, we have the following:

$$a_1 < a_2 < a_3 < \dots a_n$$

and

$$b_1 < b_2 < b_3 < \dots < b_n$$

Thus, the sum of the sequences $(a_n + b_n)$ is

$$a_1 + b_1 < a_2 + b_2 < a_3 + b_3 < \dots < a_n + b_n$$

∎

(b) If a_n and b_n are strictly increasing, then $a_n \cdot b_n$ is strictly increasing.

Let $(a_n) = \frac{-1}{n}$ and let $(b_n) = n$. Then we have the following:

$$a_n = -1, \frac{-1}{2}, \frac{-1}{3}, \frac{-1}{4}, \frac{-1}{5}, \dots$$

$$b_n = 1, 2, 3, 4, 5, \dots$$

However, since the product of these two strictly increasing sequences is

$$a_n \cdot b_n = -1, -1, -1, -1, -1, \dots$$

we have that the product is not strictly increasing and thus this statement is false.

(c) If a_n and b_n are monotonic, then $a_n + b_n$ is monotonic.

This is a false statement. Consider the following monotonic sequences:

$$a_n = (1, 2, 2, 3, 3, \dots)$$

$$b_n = (-1, -1, -2, -2, -3, \dots)$$

Their sum is the sequence $(a_n + b_n) = (0, 1, 0, 1, 0, \dots)$. This sequence is not monotonic since it oscillates between 0 and 1.

(d) If a_n and b_n are monotonic, then $a_n \cdot b_n$ is monotonic.

This statement is false. Consider the following monotonic sequences:

$$a_n = (1, 1, 2, \dots)$$

$$b_n = (1, \frac{1}{2}, \frac{1}{2}, \dots)$$

Then we have that the product of these two monotonic sequences is $a_n \cdot b_n = (1, \frac{1}{2}, 1, \dots)$, which is an oscillating sequence. Thus we have that the product of two monotonic sequences is not monotonic.

(e) If a monotone sequence is bounded, then it is convergent.

This statement is true. For proof, consult the proof of the *Monotone Convergence Theorem* (Monotone Sequence Property).

(f) If a bounded sequence is monotone, then it is convergent.

This statement is true. For proof, consult the proof of the *Monotone Convergence Theorem* (Monotone Sequence Property).

(g) If a convergent sequence is monotone, then it is bounded.

This statement is true. For proof, consult the proof of *Theorem 3.2.2*:

Theorem. A convergent sequence of real numbers is bounded.

(h) If a convergent sequence is bounded, then it is monotone.

This statement is false. Consider the sequence generated by $a_n = (-1)^n \frac{1}{n} = (-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{-1}{5}, j \dots)$. This sequence converges to 0, however since it does oscillate as it converges, we have that a_n is not monotone, yet it is bounded.