Real Analysis Homework 3

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1. Find the infimum and supremum, if they exist.

   (a) Section 2.3

   4) Let $S_4 := \{1 - \frac{(-1)^n}{n} : n \in \mathbb{N}\}$.
   $\inf S_4 = \frac{1}{2}$, $\sup S_4 = 2$

   5) a)

$$\begin{aligned} A :&= \{x \in \mathbb{R} : 2x + 5 > 0\} \\ &= \{x \in \mathbb{R} : 2x > -5\} \\ &= \{x \in \mathbb{R} : x > \frac{-5}{2}\} \end{aligned}$$

   So $\inf A$ exists. So $\inf A = \frac{-5}{2}$. But since $\nexists$ an upper bound or the upper bound of $A = \infty$, then either $\sup A = \infty$, or $\sup A = DNE$.

   b)

$$\begin{aligned} B :&= \{x \in \mathbb{R} : x + 2 \geq x^2\} \\ &= \{x \in \mathbb{R} : 0 \geq x^2 - x - 2\} \\ &= \{x \in \mathbb{R} : 0 \geq (x-2)(x+1)\} \\ &= [-1, 0] \cup [0, 2] \end{aligned}$$

   So the infimum and supremum exist. So $\inf B = -1$, and $\sup B = 2$.

   d)

$$\begin{aligned} D :&= \{x \in \mathbb{R} : x^2 - 2x - 5 < 0\} \\ &= \{x \in \mathbb{R} : (x - (1 + \sqrt{6}))(x - (1 - \sqrt{6}))\} \\ &= \{x \in \mathbb{R} : 1 - \sqrt{6} < x < 1 + \sqrt{6}\} \\ &= (1 - \sqrt{6}, 1 + \sqrt{6}) \end{aligned}$$

   So both the $\inf D$ and $\sup D$ exist. So $\inf D = 1 - \sqrt{6}$ and $\sup D = 1 + \sqrt{6}$.

   (b) $A = \{x \in \mathbb{R} : x = \frac{1}{n} + (-1)^n \text{ for } n \in \mathbb{N}\}$
   $\Rightarrow \inf A = -1$, and $\sup A = \frac{3}{2}$.

(c) $B = \{x \in \mathbb{R} : x = 2 - \dfrac{(-1)^n}{n^2} \text{ for } n \in \mathbb{N}\}$
$\Rightarrow \inf B = \frac{7}{4}$, $\sup B = 3$

2. Section 2.3

9) Let $S \subseteq \mathbb{R}$ be nonempty. Show that if $u = \sup S$, then for every number $n \in \mathbb{N}$, the number $\frac{u-1}{n}$ is not an upper bound of S, but the number $\frac{u+1}{n}$ is an upper bound of S. (The converse is also true; see Exercise 2.4.3)

Proof. Let $S \subseteq \mathbb{R}$ be nonempty. We want to show that if $u = \sup S$, then for every number $n \in \mathbb{N}$, the number $\frac{u-1}{n}$ is not an upper bound of S, but the number $\frac{u+1}{n}$ is an upper bound of S.

Let $u = \sup S$. Recall the definition of the supremum:

$$\alpha = \sup S \iff (i)\ x \leq \alpha\ \forall x \in S,\ \wedge\ (ii)\ \forall \epsilon \in S, \exists x \in S \text{ s.t. } x > \alpha - \epsilon$$

u is by definition an upper bound of S, and thus by the definition of u, $u + \frac{1}{n} > u$, thus $u + \frac{1}{n}$ is also an upper bound of S, since $u + \frac{1}{n} > u\ \forall n \in \mathbb{N}$.

Now, let $\epsilon = \frac{1}{n}$. By Lemma 2.3.4, we have that $\exists s_\epsilon \in S$ s.t. $\sup S - \epsilon < s_\epsilon < \sup S$, so

$$u - \frac{1}{n} = u - \epsilon < s_\epsilon$$

$\therefore u - \frac{1}{n}$ is not an upper bound of S. ■

10) Show that if A and B are bounded subsets of $\mathbb{R}$, then $A \cup B$ is a bounded set. Show that $\sup(A \cup B) = \sup\{\sup A, \sup B\}$.

Proof. Let $A, B \subseteq \mathbb{R}$ such that A, B are bounded. We want to show that $A \cup B$ is a bounded set, and that $\sup(A \cup B) = \sup\{\sup A, \sup B\}$.

Since A is bounded, we have that

$$\inf A \leq A \leq \sup A,$$

and since B is bounded, we have that

$$\inf B \leq B \leq \sup B$$

Let $s = \max\{|\inf A|, |\sup A|\}$, and let $t = \max\{|\inf B|, |\sup B|\}$. Let $x \in A \cup B$. Then, by the definition of union, $x \in A$ or $x \in B$.

If $x \in A$, then $|x| \leq s$.
If $x \in B$, then $|x| \leq t$.

Let $r = \max\{s, t\}$.

Then if $x \in A \cup B$, then $|x| \leq r$.

$\therefore A \cup B$ is bounded. $\square$

Now, we want to show that

$$\sup(A \cup B) = \sup\{\sup A, \sup B\}$$

Since A is bounded, $\sup A$ exists by the completeness axiom. Since B is bounded, $\sup B$ exists by the completeness axiom.

Let $w = \sup\{\sup A, \sup B\} = \max\{\sup A, \sup B\}$. Then w is an upper bound for $A \cup B$ since $w \geq |\sup A|$ and $w \geq |\sup B|$. By completeness, $\sup(A \cup B)$ exists. And $\sup(A \cup B) \leq w = \sup\{\sup A, \sup B\}$.

Let z be any upper bound for $A \cup B$. Then z is an upper bound for A and for B. So $x \leq \sup A \leq z,\ \forall a \in A$ and $x \leq \sup B \leq z,\ \forall b \in B$. So $\sup\{\sup A, \sup B\} \leq z$.

$\therefore z$ is an upper bound for $A \cup B$, choose $z = \sup(A \cup B)$. So $\sup\{\sup A, \sup B\} \leq \sup(A \cup B)$.

Then $\sup\{\sup A, \sup B\} = \sup(A \cup B)$. $\blacksquare$

3. Section 2.4

4a) *Proof.* Let S be a nonempty bounded set in $\mathbb{R}$. Let $a > 0$, and let $aS = \{as : s \in S\}$. We want to show that

$$\inf(aS) = a \inf S, \text{ and } \sup(aS) = a \sup S$$

$$\because \inf S \leq s,\ \forall s \in S,$$
$$\Rightarrow a \inf S \leq aS,\ \forall as \in aS$$

For any $\epsilon > 0, \frac{\epsilon}{s} > 0$. Then we have that $\exists s_0 \in S$ s.t. $s_0 \leq \inf S + \frac{\epsilon}{a} \Rightarrow as_0 \leq a \inf S + \epsilon$, where $as_0 \in aS$.

$\therefore \inf(aS) = a \inf S$. $\square$

Now, we want to show that $\sup(aS) = a \sup S$. By the definition of the supremum, we have that $s \leq \sup S,\ \forall s \in S \Rightarrow as \leq a \sup S,\ \forall as \in aS$. So for any $\epsilon > 0, \frac{\epsilon}{a} > 0$, we have that $\exists s' \in S$ s.t. $s' = \sup S - \frac{\epsilon}{a}$.

$\therefore \sup(aS) = a \sup S$. $\blacksquare$

5) Let S be a set of nonnegative real numbers that is bounded above, and let $T := \{x^2 : x \in S\}$. Prove that if $u = \sup S$, then $u^2 = \sup T$. Give an example that shows the conclusion may be false if the restriction against negative numbers is removed.

Proof. Let S be a set of nonnegative real numbers that is bounded above, and let $T := \{x^2 : x \in S\}$. We want to show that if $u = \sup S$, then $u^2 = \sup T$.

Suppose $u = \sup S$. Then $s \leq u,\ \forall s \in S$.
$\Rightarrow 0 \leq s \leq u$
$\Rightarrow 0 \leq s^2 \leq u^2$, because if $a, b \geq 0$ s.t. $a \leq b$, then $a^2 \leq b^2$.

So $s^2 \leq u^2, \forall s \in S \Rightarrow t \leq u^2\ \forall t \in T$.

$\therefore T$ is bounded above, where u^2 is an upper bound of T. $\square$

Thus we've satisfied one property of the supremum. Now, for the other, suppose w is an upper bound of T and $w \leq u^2$. Then $w \geq 0$, and $\sqrt{w} \leq u$, by the definition of T.

Since $u = \sup S$, we have that $\exists s_0 \in S$ s.t. $\sqrt{w} \leq s_0$.
$\Rightarrow w < s_0^2$, which contradicts the fact that w is an upper bound of T.

$\therefore \sup T = u^2$. ■

Example: Let $S := (-2, 1)$. Then $\sup S = 1$. Then $T := (1, 4)$, which yields $\sup T = 4$, and $4 \neq 1$.

8) Let X be a nonempty set, and let f and g be defined on X and have bounded ranges in $\mathbb{R}$. Show that

$$\sup\{f(x)\ +\ g(x) : x \in X\} \leq \sup\{f(x) : x \in X\}\ +\ \sup\{g(x) : x \in X\}$$

Proof. Let $A = \{f(x) : x \in X\}, B = \{g(x) : x \in X\}$, where A and B are bounded above. Let $C = \{a + b : a \in A, b \in B\}$. Since A and B are bounded, we have that $a \leq \sup A\ \forall\ a \in A$, and $b \leq \sup B\ \forall\ b \in B$. Thus we have that $a + b \in C$, by the definition of C. So $a + b \leq \sup A + \sup B\ \forall a \in A$, and $\forall b \in B$. Thus we also have that $a\ +\ b \in C$. Since $a + b \leq \sup A + \sup B \Rightarrow \sup A\ + \sup B$ is an upper bound for C. Thus by completeness and the definition of C, $\sup C \leq \sup A + \sup B$. ■

Example: Let $X = [-1, 1]$ and let $f(x) = x$ and $g(x) = -x$. Then we have $\sup\{f(x) : x \in X\} = 1$ and $\sup\{g(x) : x \in X\} = 1$. But $\{f(x) + g(x) : x \in X\} = \{x - x : x \in X\} = \{0\}$.

$$\begin{aligned}\therefore \sup\{f(x) + g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\} \\ = 2\end{aligned}$$

9a) Let $X = Y := \{x \in \mathbb{R} : 0 < x < 1\}$. Define $h : X \times Y \to \mathbb{R}$ by $h(x, y) := 2x + y$. For each $x \in X$, find $f(x) := \sup\{h(x, y) : y \in Y\}$; then find $\inf\{f(x) : x \in X\}$.
If X and Y are between 0 and 1, then the range of $f(x) = (0, 3)$, thus $\inf(f(x)) = 0$.

14) If $y > 0$, show that $\exists n \in \mathbb{N}$ such that $\frac{1}{2^n} < y$.

Proof. Let $y > 0$. By Corollary 2.4.5, $\exists n \in \mathbb{N}$ such that $0 < \frac{1}{n} < y$.
Since $n < 2^n$, we have

$$0 < \frac{1}{2^n} < \frac{1}{n} < y$$

■

4. Section 2.5

2) If $S \subseteq \mathbb{R}$ is nonempty, show that S is bounded if and only if there exists a closed bounded interval I such that $S \subseteq I$.

Proof. Let $S \subseteq \mathbb{R}$ be nonempty. We want to show that S is bounded if and only if there exists a closed, bounded interval I such that $S \subseteq I$. We prove it by cases, one for each direction of the "if and only if" condition.

Case 1: ($\Leftarrow$) Assume that there exists a closed, bounded interval I such that $S \subseteq I$; that is, define $I := [a, b]$, where $a, b \in \mathbb{R}$.

Then $\min I = a$, and $\max I = b$. Thus we have that $\forall x \in I, a \leq x$, and so a is a lower bound of I. Also, $\forall x \in I, x \leq b$, and so b is an upper bound of I. By completeness, we have that $\inf I$ and $\sup I$ exist. Specifically, we have that $\min I = \inf I = a$, and $\max I = \sup I = b$.

Since $S \subseteq I$, we know that $\forall s \in S, a \leq s \leq b$. Thus by transitivity, we have that $\because \sup I = b \Rightarrow \sup S = b$, and $\because \inf I = a \Rightarrow \inf S = a$.

$\therefore$ If there exists a closed, bounded interval I such that $S \subseteq I$, then S is bounded. □

Case 2: ($\Rightarrow$) Conversely, Assume that S is bounded. Then we have that $\exists x \in S$ s.t. $x \leq s$, $\forall s \in S$, and that $\exists y \in S$ s.t. $s \leq y$, $\forall s \in S$. Thus by completeness, $\inf S$ and $\sup S$ exist.

Let $\inf S = a$, and let $\sup S = b$. Since this holds, we can explicitly define $S := (a, b)$.

By the Archimedian property, we have that $\forall s \in S$, $\exists n \in \mathbb{N}$, s.t. $n \leq s < n + 1$.

Define an interval $I := [\lfloor a \rfloor, \lceil b \rceil]$. Thus we now have that $\lfloor a \rfloor \leq \inf S$, and that $\sup S \leq \lceil b \rceil$. Hence $S \subseteq I$.

$\therefore$ I is a closed, bounded interval by construction, such that $S \subseteq I$.

■

3) If $S \subseteq \mathbb{R}$ is a nonempty bounded set, and $I_s := [\inf S, \sup S]$, show that $S \subseteq I_s$. Moreover, if J is any closed bounded interval containing S, show that $I_s \subseteq J$.

Proof. Let $S \subseteq \mathbb{R}$ be a nonempty, bounded set, and let $I_s := [\inf S, \sup S]$. We want to show that $S \subseteq I_s$, and that if J is any closed, bounded interval that contains S, then $I_s \subseteq J$.

Let $\inf S = a$ and $\sup S = b$.

First, assume that $\nexists \min S, \max S$. Then we have that $I_s \supset S$. Since $\sup \notin S$ and $\inf \notin S$, by the definition of infimum and supremum, respectively. We know this to be the case since the only time $\inf S \in S$ is if $\exists \min S$, and also $\sup S \in S$ if $\exists \max S$. But since $\inf S \in I_s$, and $\sup S \in I_s$, by the definition of I_s, we have that $S \subset I_s$.

Now suppose that $\sup S, \inf S \in S$. Then $I_s = S$, since the bounds are the same. That is, let $\inf S = \alpha$, and let $\sup S = \beta$. Then $S = I_s \iff S := [\alpha, \beta]$. This is because $\min S = \inf S = \alpha$ and $\max S = \sup S = \beta$.

$\therefore\ S \subseteq I_s$.

Now, let J be a nonempty, bounded, closed set such that $S \subseteq J$. We want to show that $I_s \subseteq J$. Since J is bounded, we can define $J := [a, b]$, where $a, b \in \mathbb{R}$. Similarly as was to be shown above, we know that $\min J = a$, and $\max J = b$. So, we know that $\inf J = \min J = a$, and $\sup J = \max J = b$. Since $S \subseteq J$, we know that if $S \subsetneq J$,

i. $\inf S \notin S$ but $\inf S \in J$, and
ii. $\sup S \notin S$ but $\sup S \in J$

Thus since $\inf S, \sup S \in J, I_s \subseteq J$, since $\inf S, \sup S \in I_s$. Also, if $S = I_s$, then clearly $I_s \subseteq J$. ■

5. Prove that for every $x \in \mathbb{R}$ and for each $n \in \mathbb{N}$, there exists a rational number r_n such that $|x - r_n| < \frac{1}{n}$.

Proof. Let $x \in \mathbb{R}$, and let $n \in \mathbb{N}$. Then we have $x - \frac{1}{n} < x + \frac{1}{n}$. So $x - \frac{1}{n}, x + \frac{1}{n} \in \mathbb{R}$. By Theorem 2.4.8, we have that $\exists r_n \in \mathbb{Q}$ s.t. $x - \frac{1}{n} < r_n < \frac{1}{n} \Rightarrow \frac{-1}{n} < r_n - x < \frac{1}{n}$.

So $|r_n - x| < \frac{1}{n}$ and $|x - r_n| < \frac{1}{n}$. ■

6. A *dyadic rational* is a number of the form $\frac{k}{2^n}$ for some $k, n \in \mathbb{Z}$. Prove that if $a, b \in \mathbb{R}$ and $a < b$, then there exists a dyadic rational q such that $a < q < b$.

Proof. Let $a, b \in \mathbb{R}$ such that $a < b$. We want to show that $\exists q = \frac{k}{n}$ s.t. $a < q < b$.

By question 14 from Section 2.4, we know that $\forall y > 0, \exists n$ s.t. $\frac{1}{2^n} < y$. By the Archimedian property, we have $0 < \frac{1}{2^n} < \frac{1}{n} < y$.

Case 1: Let $a > 0$. So $0 < a < b$. By the Archimedian property again, $\exists n \in \mathbb{N}$ s.t. $0 < \frac{1}{2^n} < \frac{1}{n} < b - a$. So $\frac{1}{2^n} < b - a$. So $1 + a * 2^n < b * 2^n$. By the Archimedian property again, since $a * 2^n > 0$, $\exists m \in \mathbb{N}$ s.t. $m - 1 \leq a * 2^n < m$. So $m \leq a * 2^n + 1 < m + 1$.

Now, combine $a * 2^n < m \leq a * 2^n + 1 < b * 2^n$. So $a < \frac{m}{2^n} < b$, and $q = \frac{m}{2^n}$. $\square$

Case 2: If $a \leq 0$, choose $p \in \mathbb{Z}$ s.t. $p \geq |a|$. Apply Case 1 to $0 < a + p < b + p$ to get $a + p < \frac{m}{2^n} < b + p$. So $a < \frac{m}{2^n} - p < b$. So $a < q < b$, where $q = \frac{m - 2^n * p}{2^n} = \frac{k}{2^n}$. $\blacksquare$

7. Prove, if true. Provide a counterexample if false.

   (a) If A and B are nonempty, bounded subsets of $\mathbb{R}$, then $\sup(A \cap B) \leq \sup A$.

   Proof. Let A and B be nonempty, bounded subsets of $\mathbb{R}$. We want to show that $\sup(A \cap B) \leq \sup A$.

   Consider the case where $A \cap B = \emptyset$. Then $\sup(\emptyset) = -\infty$. Since $A, B \subseteq \mathbb{R}$ and $A, B \not\subseteq \overline{\mathbb{R}}$, we have that since A is nonempty, $A \neq \{-\infty\} \Rightarrow \sup A \neq -\infty$. Thus if $A \cap B = \emptyset \Rightarrow \sup(A \cap B) < \sup A$.

   Now, consider the case where $A \cap B \neq \emptyset$.

   By the definition of intersection, $\sup(A \cap B) = \sup A \iff A \cap B = A$. Also by the definition of intersection, we have that if $A \cap B \neq A$, then $A \cap B \subset A$ and $A \cap B \subset B$. This means that it's impossible to have a set after the intersection that is larger than both A and B. This implies that the resulting set will yield $\sup(A \cap B) < \sup A$, since $\sup(A \cap B) = \sup A \Rightarrow A \cap B = A$.

   $\therefore \sup(A \cap B) \leq \sup A$. $\blacksquare$

   (b) If $A + B = \{a + b : a \in A, b \in B\}$, where A and B are nonempty, bounded subsets of $\mathbb{R}$, then $\sup(A + B) = \sup A + \sup B$.

   Proof. Let A and B be nonempty bounded subsets of $\mathbb{R}$, and let $A + B = \{a + b : a \in A, b \in B\}$. We want to show that $\sup(A + B) = \sup A + \sup B$.

   Since A and B are bounded, we know that $\sup A$ and $\sup B$ exist, and that $x \leq \sup A$, $\forall x \in A$, and $y \leq \sup B$, $\forall y \in B$. So $x + y \in A + B$ and $x + y \leq \sup A + \sup B$, $\forall x \in A, \forall y \in B$. Then by completeness, $\sup(A + B) \leq \sup A + \sup B$. $\square$

   Now we must show that $\sup A + \sup B \leq \sup(A + B)$.

   Let $y \in B$ be fixed. Since $x + y \leq \sup(A + B)$, then $x \leq \sup(A + B) - y$, $\forall x \in A$.

So $\sup(A+B)-y$ is an upper bound for A. By completeness, we have that $\sup A \leq \sup(A+B)-y$. Then $y \leq \sup(A+B)-\sup A$. This is true for all $y \in B$.

So $\sup(A+B)-\sup A$ is an upper bound for B, and $\sup B \leq \sup(A+B)-\sup A$.

$\therefore \sup A+\sup B \leq \sup(A+B)$. ■

(c) If $A-B=\{a-b : a \in A, b \in B\}$, where A and B are nonempty, bounded subsets of $\mathbb{R}$, then $\sup(A-B)=\sup A-\sup B$.

Counterexample: Let $A := [-2,0]$ and let $B := [1,4]$. Then we have that $A-B := [-4,-3]$. Then we have the following:

$$\begin{aligned}\sup(A-B) &= \sup A-\sup B\\ \sup([-4,-3]) &= \sup([-2,0])-\sup([1,4])\\ -3 &= 0-4\\ -3 &\neq -4\end{aligned}$$

Thus $\sup(A-B) \neq \sup A-\sup B$.