Real Analysis Homework 11

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1. Give an example of nonconstant functions f and g such that $(fg)' = f'g'$.

Let $f(x) = e^{2x} = g(x)$. Then

$$\begin{aligned}(fg)' &= f'g' \\ f'(x)g(x) + f(x)g'(x) &= f'(x)g'(x) \\ 2e^{2x} \cdot e^{2x} + e^{2x} \cdot 2e^{2x} &= 2e^{2x} \cdot 2e^{2x} \\ 2e^{4x} + 2e^{4x} &= 4e^{4x} \\ 4e^{4x} &= 4e^{4x}\end{aligned}$$

Thus we have that $(fg)' = f'g'$.

2. Suppose that f is differentiable at 2 and 4 with $f(2) = 2$, $f(4) = 3$, $f'(2) = \pi$, and $f'(4) = e$.

(a) If $g(x) = xf(x^2)$, find $g'(2)$.

$$\begin{aligned}g'(x) &= (xf(x^2))' \\ &= x' \cdot f(x^2) + x \cdot f'(x^2) \cdot 2x \qquad \text{by the } \textit{Product Rule} \text{ and the } \textit{Chain Rule} \\ &= 1 \cdot f(x^2) + x \cdot f'(x^2) \cdot 2x \\ &\Downarrow \\ g'(2) &= f(x^2) + x \cdot f'(x^2) \cdot 2x \\ &= f(2^2) + 2 \cdot f'(2^2) \cdot 2(2) \\ &= f(4) + 2 \cdot f'(4) \cdot 2(2) \\ &= 3 + 2 \cdot e \cdot 4 \\ &= 3 + 8e\end{aligned}$$

So $g'(2) = 3 + 8e$.

(b) If $g(x) = f^2(\sqrt{x})$, find $g'(4)$.

$$\begin{aligned}
g'(x) &= ([f(\sqrt{x})]^2)' \\
&= 2[f(\sqrt{x})] \cdot f'(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} && \text{by the } \textit{Chain Rule} \\
&\Downarrow \\
g'(4) &= 2[f(\sqrt{4})] \cdot f'(\sqrt{4}) \cdot \frac{1}{2\sqrt{4}} \\
&= 2 \cdot f(2) \cdot f'(2) \cdot \frac{1}{2 \cdot 2} \\
&= 2 \cdot 2 \cdot \pi \cdot \frac{1}{4} \\
&= \frac{4\pi}{4} \\
&= \pi
\end{aligned}$$

So $g'(4) = \pi$.

(c) If $g(x) = x/f(x^3)$, find $g'(\sqrt[3]{2})$.

$$\begin{aligned}
g'(x) &= \left(\frac{x}{f(x^3)}\right)' \\
&= \frac{x' \cdot f(x^3) - x \cdot f'(x^3) \cdot 3x^2}{\left(f(x^3)\right)^2} && \text{by the } \textit{Quotient Rule} \\
&= \frac{1 \cdot f(x^3) - x \cdot f'(x^3) \cdot 3x^2}{(f(x^3))^2} \\
&\Downarrow \\
g'(\sqrt[3]{2}) &= \frac{f((\sqrt[3]{2})^3) - \sqrt[3]{2} \cdot f'((\sqrt[3]{2})^3) \cdot 3(\sqrt[3]{2})^2}{(f((\sqrt[3]{2})^3))^2} \\
&= \frac{f(2) - 3(\sqrt[3]{2})^2 \sqrt[3]{2} \cdot f'(2)}{(f(2))^2} \\
&= \frac{2 - 3 \cdot 2 \cdot \pi}{2^2} \\
&= \frac{2 - 6 \cdot \pi}{4} \\
&= \frac{1 - 3 \cdot \pi}{2}
\end{aligned}$$

So $g'(\sqrt[3]{2}) = \frac{1-3\pi}{2}$.

3. Determine if each function is differentiable at the given point. If so, find its derivative. If not, explain why not.

(a) At $x = 1$ for $f(x) = \begin{cases} 3x - 2 & \text{if } x < 1 \\ x^3 & \text{if } x \geq 1 \end{cases}$

We note that $f(1) = 1^3 = 1$. First, let us find $f'(x)$ for each part of the function. So when $x < 1$:

$$\begin{aligned} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{3(x+h) - 2 - (3x - 2)}{h} \\ &= \lim_{h \to 0} \frac{3x + 3h - 2 - 3x + 2}{h} \\ &= \lim_{h \to 0} \frac{3h}{h} \\ &= \lim_{h \to 0} 3 \\ &= 3 \end{aligned}$$

So $f'(x) = 3$ when $x < 1$ And when $x \geq 1$:

$$\begin{aligned} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h \to 0} \frac{h^3 - 3h^2x + 3hx^2 + x^3 - x^3}{h} \\ &= \lim_{h \to 0} \frac{h^3 - 3h^2x + 3hx^2}{h} \\ &= \lim_{h \to 0} \frac{h(h^2 - 3hx + 3x^2)}{h} \\ &= \lim_{h \to 0} h^2 - 3hx + 3x^2 \\ &= 0^2 + 3(0)x + 3x^2 \\ &= 0 + 0 + 3x^2 \\ &= 3x^2 \end{aligned}$$

So $f'(x) = 3x^2$ when $x \geq 1$.

Now we must check for differentiability when $x = 1$. So

$$\begin{aligned} \lim_{x \to c} \frac{f(x) - f(c)}{x - c} &= \lim_{x \to 1^+} \frac{x^3 - 1}{x - 1} \\ &= \lim_{x \to 1^+} \frac{\cancel{(x-1)}(x^2 + x + 1)}{\cancel{(x-1)}} \\ &= \lim_{x \to 1^+} (x^2 + x + 1) \\ &= 1^2 + 1 + 1 \\ &= 1 + 1 + 1 \\ &= 3 \end{aligned}$$

And

$$\begin{aligned}
\lim_{x\to c}\frac{f(x)-f(c)}{x-c} &= \lim_{x\to 1^-}\frac{3x-2-1}{x-1} \\
&= \lim_{x\to 1^-}\frac{3x-3}{x-1} \\
&= \lim_{x\to 1^-}\frac{3\cancel{(x-1)}}{\cancel{(x-1)}} \\
&= \lim_{x\to 1^-} 3 \\
&= 3
\end{aligned}$$

So since $\lim\limits_{x\to 1^-}\frac{f(x)-f(c)}{x-c} = 3 = \lim\limits_{x\to 1^+}\frac{f(x)-f(c)}{x-c}$, we have that the limits are equal, and thus the limit exists which yields that f is differentiable at $x = 1$.

(b) At $x = 1$ for $f(x) := \begin{cases} 2x+1 & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1 \end{cases}$

In order for f to be differentiable, it must also be continuous. So let us first ensure that f is continuous.

So we have

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2x+1 = 2(1)+1 = 3$$

and

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 = 1^2 = 1$$

Since $\lim\limits_{x\to 1^-} f(x) = 3 \neq 1 = \lim\limits_{x\to 1^+} f(x)$, we have that f is not continuous, and thus f is not differentiable at $x = 1$.

(c) At $x = 1$ for $f(x) := \begin{cases} 3x-2 & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1 \end{cases}$

We will show that f is **not** differentiable at $x = 1$. We first note that $f(1) = 1^2 = 1$. Now let us find the derivatives of f for each part of the piece wise function. So for $x < 1$:

$$\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{3(x+h)-2-(3x-2)}{h} \\
&= \lim_{h\to 0}\frac{3x+3h-2-3x+2}{h} \\
&= \lim_{h\to 0}\frac{3h}{h} \\
&= \lim_{h\to 0} 3 \\
&= 3
\end{aligned}$$

So $f'(x) = 3$ when $x < 1$.

And for the case where $x \geq 1$:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{h^2 + 2hx + x^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{h(h+2x)}{h} \\
&= \lim_{h\to 0} (h+2x) \\
&= 0 + 2x \\
&= 2x
\end{aligned}$$

So $f'(x) = 2x$ for $x \geq 1$.

Now we must show that f is **not** differentiable at $x = 1$. So

$$\begin{aligned}
\lim_{x\to c} \frac{f(x)-f(c)}{x-c} &= \lim_{x\to 1^-} \frac{3x-2-1}{x-1} \\
&= \lim_{x\to 1^-} \frac{3x-3}{x-1} \\
&= \lim_{x\to 1^-} \frac{3\cancel{(x-1)}}{\cancel{x-1}} \\
&= \lim_{x\to 1^-} 3 \\
&= 3
\end{aligned}$$

And

$$\begin{aligned}
\lim_{x\to c} \frac{f(x)-f(c)}{x-c} &= \lim_{x\to 1^+} \frac{x^2-1}{x-1} \\
&= \lim_{x\to 1^+} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} \\
&= \lim_{x\to 1^+} (x+1) \\
&= 1+1 \\
&= 2
\end{aligned}$$

So since $\lim_{x\to 1^-} f(x) = 3 \neq 2 = \lim_{x\to 1^+} f(x)$, we have that the limit does not exist at $x = 1$, and thus f is not differentiable at $x = 1$, but is continuous.

(d) (Sec. 6.1, pr. 4) At $x = 0$ for $f(x) := \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

First, we note that $0 \in \mathbb{Q}$ and $f(0) = 0^2 = 0$.

Let $g(x) := \frac{f(x)-f(0)}{x-0} = \frac{f(x)}{x}$. Then for $x \neq 0$:

$$g(x) := \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

So, we have that the given function is differentiable at 0 if and only if $\lim\limits_{x\to 0} g(x)$ exists. In this case, the limit is $f'(0)$.

We note that $-|x| \leq g(x) \leq |x|$, and that $\lim\limits_{x\to 0} -|x| = \lim\limits_{x\to 0} |x| = 0$.

Recall the *Squeeze Theorem*:

Theorem 1 (Squeeze Theorem). Let $A \subseteq \mathbb{R}$, let $f, g, h : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. If

$$f(x) \leq g(x) \leq h(x) \quad \text{for all} \quad x \in A,\ x \neq c,$$

and if $\lim\limits_{x\to c} f = L = \lim\limits_{x\to c} h$, then $\lim\limits_{x\to c} g = L$.

By the *Squeeze Theorem* $\lim\limits_{x\to 0} g(x) = 0$. Thus we have that f is differentiable at $x = 0$ and $f'(0) = 0$.

(e) At $x = 0$ for $f(x) := \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

We want to show that $f(x)$ is **not** differentiable at $x = 0$. First we note that $f(0) = 0$. So let us first find the derivatives of $f(x)$ for each piece of the function. So for $x \in \mathbb{Q}$:

$$\begin{aligned} \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{x+h-x}{h} \\ &= \lim_{h\to 0} \frac{h}{h} \\ &= \lim_{h\to 0} 1 \\ &= 1 \end{aligned}$$

So $f'(x) = 1$ when $x \in \mathbb{Q}$.

Now, for $x \in \mathbb{R} \setminus \mathbb{Q}$:

$$\begin{aligned} \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{0-0}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0 \end{aligned}$$

So $f'(x) = 0$ when $x \in \mathbb{R} \setminus \mathbb{Q}$.

Now we want to show that f is **not** differentiable at $x = 0$. So there's four cases we must consider: x approaches zero from the left along a sequence of rational numbers, x approaches zero from the right along a sequence of rational numbers, x approaches zero from the left along a sequence of irrational numbers, and x approaches zero from the right along a sequence of irrational numbers.

For the case where x approaches 0 from the left along a sequence of rational numbers:

$$\begin{aligned}\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c} &= \lim_{x\in\mathbb{Q}\to 0^-}\frac{f(x)-f(0)}{x-0}\\ &= \lim_{x\in\mathbb{Q}\to 0^-}\frac{x-0}{x}\\ &= \lim_{x\in\mathbb{Q}\to 0^-}\frac{x}{x}\\ &= \lim_{x\in\mathbb{Q}\to 0^-} 1\\ &= 1\end{aligned}$$

And for x approaches 0 from the right along a sequence of rational numbers:

$$\begin{aligned}\lim_{x\to c^+}\frac{f(x)-f(c)}{x-c} &= \lim_{x\in\mathbb{Q}\to 0^+}\frac{f(x)-f(0)}{x-0}\\ &= \lim_{x\in\mathbb{Q}\to 0^+}\frac{x-0}{x}\\ &= \lim_{x\in\mathbb{Q}\to 0^+}\frac{x}{x}\\ &= \lim_{x\in\mathbb{Q}\to 0+} 1\\ &= 1\end{aligned}$$

For the case where x approaches 0 from the left along a sequence of irrational numbers:

$$\begin{aligned}\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c} &= \lim_{x\in\mathbb{R}\setminus\mathbb{Q}\to 0^-}\frac{f(x)-f(0)}{x-0}\\ &= \lim_{x\in\mathbb{R}\setminus\mathbb{Q}\to 0^-}\frac{0-0}{x}\\ &= \lim_{x\in\mathbb{R}\setminus\mathbb{Q}\to 0^-}\frac{0}{x}\\ &= \lim_{x\in\mathbb{R}\setminus\mathbb{Q}\to 0^-} 0\\ &= 0\end{aligned}$$

And for x approaching 0 from the right along a sequence of irrational numbers:

$$\begin{aligned}\lim_{x\to c^+}\frac{f(x)-f(c)}{x-c} &= \lim_{x\in\mathbb{R}\setminus\mathbb{Q}\to 0^+}\frac{f(x)-f(0)}{x-0}\\ &= \lim_{x\in\mathbb{R}\setminus\mathbb{Q}\to 0^+}\frac{0-0}{x}\\ &= \lim_{x\in\mathbb{R}\setminus\mathbb{Q}\to 0^+}\frac{0}{x}\\ &= \lim_{x\in\mathbb{R}\setminus\mathbb{Q}\to 0^+} 0\\ &= 0\end{aligned}$$

So we have that

$$\lim_{x\in\mathbb{Q}\to 0^-}\frac{f(x)-f(0)}{x} = \lim_{x\in\mathbb{Q}\to 0^+}\frac{f(x)-f(0)}{x} = 1$$

and

$$0 = \lim_{x\in\mathbb{R}\setminus\mathbb{Q}\to 0^+}\frac{f(x)-f(0)}{x} = \lim_{x\in\mathbb{R}\setminus\mathbb{Q}\to 0^-}\frac{f(x)-f(0)}{x}$$

However, since these limits are not equal to each other since $0 \neq 1$, we have that the limit does not exist at $x = 0$, and thus f is **not** differentiable at $x = 0$ but is continuous.

4. (a) Prove: If f is differentiable on $\mathbb{R}$, then $f'(x) = \lim\limits_{h\to 0}\frac{f(x)-f(x-h)}{h}$

Proof. Assume that $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$. Then we know that

$$f'(x) := \lim_{x\to c}\frac{f(x)-f(c)}{x-c} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Recall the definition of the derivative:

Definition 1. Let $I \subseteq \mathbb{R}$ be an interval, let $f : I \to \mathbb{R}$, and let $c \in I$. We say that a real number L is the **derivative of f at c** if given any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $x \in I$ satisfies $0 < |x - c| < \delta(\varepsilon)$, then

$$\left|\frac{f(x)-f(c)}{x-c} - L\right| < \varepsilon.$$

In this case we say that f is **differentiable** at c, and we write $f'(c)$ for L. In other words, the derivative of f at c is given by the limit

$$f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

provided this limit exists. (We allow the possibility that c may be the endpoint of the interval.)

Assume $f'(x) = L$ for some $L \in \mathbb{R}$. Then

$$\forall\, \varepsilon > 0,\ \exists\, \delta(\varepsilon) > 0 \text{ s.t. } [x \in A \text{ s.t. } 0 < |x - c| < \delta(\varepsilon)] \implies \left|\frac{f(x) - f(c)}{x - c} - L\right| < \varepsilon$$

Now, let $h := x - c$. Then $c = x - h$. So

$$\forall\, \varepsilon > 0,\ \exists\, \delta(\varepsilon) > 0 \text{ s.t. } [x \in A \text{ s.t. } 0 < |h| < \delta(\varepsilon)] \implies \left|\frac{f(x) - f(x-h)}{h} - L\right| < \varepsilon$$

$$\equiv$$

$$\lim_{h \to 0} \frac{f(x) - f(x-h)}{h} = L$$

And thus

$$f'(x) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h}$$

■

(b) Show $f'(x) = \lim_{h \to 0} \frac{3f(x+h) - f(x) - 2f(x-h)}{5h}$

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{3f(x+h) - f(x) - 2f(x-h)}{5h} \\
&= \frac{3}{5} \lim_{h \to 0} \frac{f(x+h)}{h} - \frac{1}{5} \lim_{h \to 0} \frac{f(x)}{h} - \frac{2}{5} \lim_{h \to 0} \frac{f(x-h)}{h} \\
&= \frac{3}{5} \lim_{h \to 0} \frac{f(x+h)}{h} - \frac{3}{5} \lim_{h \to 0} \frac{f(x)}{h} + \frac{2}{5} \lim_{h \to 0} \frac{f(x)}{h} - \frac{2}{5} \lim_{h \to 0} \frac{f(x-h)}{h} \\
&= \frac{3}{5} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \frac{2}{5} \lim_{h \to 0} \frac{f(x) - f(x-h)}{h} \\
&= \frac{3}{5} \cdot f'(x) + \frac{2}{5} \cdot f'(x) \\
&= f'(x)
\end{aligned}$$

Thus $f'(x) = \lim_{h \to 0} \frac{3f(x+h) - f(x) - 2f(x-h)}{5h}$.

(c) Find constant c such that $f'(x) = \lim_{h \to 0} \frac{5f(x+h) - f(x) - 4f(x-h)}{ch}$

Let $c = 9$. Then we want to show that

$$f'(x) = \lim_{h \to 0} \frac{5f(x+h) - f(x) - 4f(x-h)}{9h}$$

So

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{5f(x+h) - f(x) - 4f(x-h)}{9h} \\
&= \frac{5}{9}\lim_{h\to 0}\frac{f(x+h)}{h} - \frac{1}{9}\lim_{h\to 0}\frac{f(x)}{h} - \frac{4}{9}\lim_{h\to 0}\frac{f(x-h)}{h} \\
&= \frac{5}{9}\lim_{h\to 0}\frac{f(x+h)}{h} - \frac{5}{9}\lim_{h\to 0}\frac{f(x)}{h} + \frac{4}{9}\lim_{h\to 0}\frac{f(x)}{h} - \frac{4}{9}\lim_{h\to 0}\frac{f(x-h)}{h} \\
&= \frac{5}{9}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \frac{4}{9}\lim_{h\to 0}\frac{f(x)-f(x-h)}{h} \\
&= \frac{5}{9}\cdot f'(x) + \frac{4}{9}\cdot f'(x) \\
&= f'(x)
\end{aligned}
$$

Thus $f'(x) = \lim\limits_{h\to 0} \frac{5f(x+h)-f(x)-4f(x-h)}{ch}$ for $c = 9$.

5. Prove, if true or provide a counterexample, if false.
For all parts: Let A be an interval, $c \in A$ and $f, g : A \to \mathbb{R}$.

(a) If f and g are differentiable at c, then $f + g$ is differentiable at c.

This is true by *Theorem 6.1.3*:

Theorem 2. Let $I \subseteq \mathbb{R}$ be an interval, let $c \in I$, and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be functions that are differentiable at c. Then:

i. If $\alpha \in \mathbb{R}$, then the function αf is differentiable at c, and

$$(\alpha f)'(c) = \alpha f'(c)$$

ii. The function $f + g$ is differentiable at c, and

$$(f + g)'(c) = f'(c) + g'(c)$$

iii. (Product Rule) The function fg is differentiable at c, and

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

iv. (Quotient Rule) If $g(c) \neq 0$, then the function f/g is differentiable at c, and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$$

(b) If f and g are differentiable at c, then $g \circ f$ is differentiable at c.

This is a false statement. Consider $f(x) := x + 1$. Then $f(c) = c + 1$.

Let $g(x) := \begin{cases} 1, & x \leq c+1 \\ 4, & x > c+1 \end{cases}$

Then we have that $g'(c+1)$ does not exist, and thus g is not differentiable at $f(c)$.

(c) If $f = g^2$ and f is differentiable on (a, b), then g is differentiable on (a, b).

This is a false statement. Consider $g^2 : (0, 1) \to \mathbb{R}$ given by $g(x) := -1$. Then $f : (0, 1) \to \mathbb{R}$ is $f(x) := -1$, which is differentiable on $(0, 1)$. However, g is not differentiable on $(0, 1)$ since $g(x) := \sqrt{-1} = i \notin \mathbb{R}$, and thus g is not differentiable on $(0, 1)$ since $(0, 1) \subset \mathbb{R}$, and $i \notin (0, 1)$.