Real Analysis Homework 10

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1. Section 5.4

7. If $f(x) := x$ and $g(x) := \sin x$, show that both f and g are uniformly continuous on $\mathbb{R}$, but that their product fg is not uniformly continuous on $\mathbb{R}$.

Proof. We want to show that $f(x) := x$ is uniformly continuous on $\mathbb{R}$.

In the interest of being explicit, recall the definitions of continuity and uniform continuity, respectively, and note their differences if $f : A \subseteq \mathbb{R} \to \mathbb{R}$:

Continuity:

$$\forall\ x \in A\ \forall\ \varepsilon > 0\ \exists\ \delta > 0 \text{ s.t. } \forall\ y \in A;\ |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

Uniform Continuity:

$$\forall\ \varepsilon > 0\ \exists\ \delta > 0 \text{ s.t. } \forall\ x, y \in A;\ |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

We note that the difference is that for continuity, one takes an arbitrary point $x \in A$, and thus there must exist a distance δ, whereas for uniform continuity, we have that a single δ must work uniformly for all points x and y.

So, $\forall\ x, u \in \mathbb{R}$, we have the following:

$$|f(x) - f(u)| = |x - u| < \varepsilon$$

So, let $\delta = \varepsilon$.

$\therefore\ f(x) := x$ is uniformly continuous on $\mathbb{R}$. ■

Proof. Now we want to show that $g(x) := \sin x$ is uniformly continuous.

So, $\forall\ x, y \in \mathbb{R}$:

$$\begin{aligned}
|g(x) - g(u)| &= |\sin x - \sin u| \\
&= \left|2\cos\left(\frac{x+u}{2}\right)\sin\left(\frac{x-u}{2}\right)\right| \\
&= 2\left|\cos\left(\frac{x+u}{2}\right)\sin\left(\frac{x-u}{2}\right)\right| \\
&\leq 2\left|\sin\left(\frac{x-u}{2}\right)\right| && \because\ |\cos(x)| \leq |x|\ \forall\ x \in \mathbb{R} \\
&\leq 2\left|\frac{x-u}{2}\right| \\
&= \frac{2}{2}|x-u| \\
&= |x-u| \\
&< \varepsilon
\end{aligned}$$

So if we choose $\delta = \varepsilon$, we have that $g(x)$ is uniformly continuous.

■

Now, we want to show that fg is not uniformly continuous on $\mathbb{R}$. To do this, recall the *Nonuniform Continuity Criteria*:

Theorem (**Nonuniform Continuity Criteria**). Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Then the following statements are equivalent:

i. f is not uniformly continuous on A.

ii. There exists an $\varepsilon_0 > 0$ such that for every $\delta > 0$ there are points x_δ, u_δ in A such that $|x_\delta - u_\delta| < \delta$ and $|f(x_\delta) - f(u_\delta)| \geq \varepsilon_0$.

iii. There exists an $\varepsilon_0 > 0$ and two sequences (x_n) and (u_n) in A such that $\lim(x_n - u_n) = 0$ and $|f(x_n) - f(u_n)| \geq \varepsilon_0 = 1$ for all $n \in \mathbb{N}$.

So, let $(x_n), (u_n) \subseteq \mathbb{R}$ be given by $x_n := 2n\pi$, and $u_n := 2n\pi + \frac{1}{n}$ for $n \in \mathbb{N}$. Then we have that

$$(x_n - u_n) = 2n\pi - (2n\pi + \frac{1}{n}) = -\frac{1}{n}$$

and thus

$$\lim_{n\to\infty}(x_n - u_n) = \lim_{n\to\infty} -\frac{1}{n} = 0$$

Now we have the following:

$$\begin{aligned}
|(fg)(x_n) - (fg)(u_n)| &= \left|2n\pi \sin(2n\pi) - \left(2n\pi + \frac{1}{n}\right) \sin\left(2n\pi + \frac{1}{n}\right)\right| \\
&= \left|2n\pi \cdot 0 - \left(2n\pi + \frac{1}{n}\right) \sin\left(2n\pi + \frac{1}{n}\right)\right| \\
&= \left|-\left(2n\pi + \frac{1}{n}\right) \sin\left(2n\pi + \frac{1}{n}\right)\right| \\
&= \left(2n\pi + \frac{1}{n}\right) \sin\left(2n\pi + \frac{1}{n}\right) \\
&= \left(2n\pi + \frac{1}{n}\right) \sin\left(\frac{1}{n}\right) && \because \sin(2n\pi) = 0 \ \forall\ n \in \mathbb{N} \\
&= 2n\pi \sin\left(\frac{1}{n}\right) + \left(\frac{1}{n}\right) \sin\left(\frac{1}{n}\right)
\end{aligned}$$

Note that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ and $\left(\sin\left(\frac{1}{n}\right)\right)$ is a bounded sequence in $\mathbb{R}$. Then

$$\lim_{n\to\infty} \frac{1}{n} \sin \frac{1}{n} = 0$$

and

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \ \forall\ x \in \mathbb{R}$$

This yields

$$\lim_{n\to\infty} 2n\pi \sin\left(\frac{1}{n}\right) = 2\pi \lim_{\frac{1}{n}\to 0} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 2\pi$$

Thus $\lim\limits_{n\to\infty} ((fg)(x_n) - (fg)(u_n)) = 2\pi$.

Let $\varepsilon = \pi$. Then $\exists\ k \in \mathbb{N}$ s.t. $\forall\ n \geq k$:

$$\begin{gathered}
2\pi - \varepsilon < (fg)(x_n) - (fg)(u_n) < 2\pi + \varepsilon \\
(fg)(x_n) - (fg)(u_n) > \pi \\
|(fg)(x_n) - (fg)(u_n)| > \pi \\
|(fg)(x_{n+k}) - (fg)(u_{n+k})| > \pi
\end{gathered}$$

Now, for $\varepsilon_0 = \pi$ and two sequences $(x_{n+k}), (u_{n+k})$, by the *Nonuniform Continuity Criteria*, (fg) is not uniformly continuous on $\mathbb{R}$.

10. Prove that if f is uniformly continuous on a bounded subset A of $\mathbb{R}$, then f is bounded on A.

Proof. Let f be uniformly continuous on a bounded subset A of $\mathbb{R}$. We want to show that f is bounded on A.

Since f is uniformly continuous, we know that $\forall\ \varepsilon > 0,\ \exists\ \delta > 0$ s.t. $\forall\ x, y \in A;\ |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$. We must find a constant $M > 0$ s.t. $|f(x)| \leq M\ \forall\ x \in A$.

Let $\varepsilon > 0$ be given, and let $\delta > 0$ also be given.

Recall the definitions of a cover, open cover, subcover, and compactness:

Definition 1. Let A be a subset of a topological space X. Let $\mathscr{U} = \{U_\alpha \cup_{\alpha \in J} U_\alpha\}$ be a collection of subsets of X. We say that $\mathscr{U}$ is a **cover** of A if $A \subseteq \cup_{\alpha \in J} U_\alpha$. $\mathscr{U}$ is called an **open cover** of A if $\mathscr{U}$ is a cover of A and each element of $\mathscr{U}$ is open. If $\mathscr{U}$ is a cover of A, then any subset of $\mathscr{U}$ that is also a cover of A is called a **subcover** of A. A space X is **compact** if every open cover of X has a finite subcover.

Also recall the following theorem:

Theorem. A compact subset of a metric space is bounded.

Since A is a subset of $\mathbb{R}$ under the usual metric, defined $d(x, y) := |x - y|$, we have that A is compact. Thus by the definition of compactness, we know that every open cover of A has a finite subcover. In particular, we know that there are finitely many open balls of radius δ. Thus

$$\exists n \in \mathbb{N} \text{ s.t. } A \subseteq \cup_{i=0}^{n} B_d(x_i, \delta),\ x_i \in A$$

Recall the definition of an open ball:

Definition 2. Let (M, d) be a metric space. Given $x \in M$ and a positive real number ε, the **open ball** centered at x with radius ε is

$$B_d(x, \varepsilon) := \{y \in M \mid d(x, y) < \varepsilon\}$$

Now, define m and M as follows:

$$m := \min_{1 \leq i \leq n} f(x_i),\ M := \max_{1 \leq i \leq n} f(x_i)$$

Let $x \in A$ be arbitrary. Since A is covered with balls $B_d(x_i, \delta)$, we know that $\exists\ x_j \in A$ s.t. $x \in B_d(x_j, \delta)$. So,

$$x \in B_d(x_j, \delta) \implies |x - x_j| < \delta \implies |f(x) - f(x_j)| < \varepsilon \iff -\varepsilon < f(x) - f(x_j) < \varepsilon$$

By the definition of M, we know that $f(x_j) \leq M$, and thus $f(x) \leq M + \varepsilon$.

By the definition of m, we know that $f(x_j) \geq m$, and thus $f(x) \geq m - \varepsilon$.

$\therefore\ \forall\ x \in A$,

$$f(x) \in [m - \varepsilon, M + \varepsilon]$$

$\therefore\ f$ is bounded on A.

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Alternative Proof:

Proof. By way of contradiction, assume that $f(A)$ is unbounded. Then we know that there exists a sequence $x_n \in A$ s.t. $|f(x_n)| \geq n \; \forall \; n$.

Since (x_n) is bounded, we know that there exists a convergent subsequence $x_{n_k} \in A$ s.t. $x_{n_k} \to x$. Now, since f is uniformly continuous, we know that $f(x_{n_k}) \to f(x)$, which contradicts the fact that $f(A)$ is unbounded. Thus we have that if f is uniformly continuous on A, then $f(A)$ is bounded as well. ∎

14. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **periodic** on $\mathbb{R}$ if there exists a number $p > 0$ such that $f(x+p) = f(x)$ for all $x \in \mathbb{R}$. Prove that a continuous periodic function on $\mathbb{R}$ is bounded and uniformly continuous on $\mathbb{R}$.

Proof. Let f be a continuous periodic function on $\mathbb{R}$. Then we know that $\exists \; p > 0$ s.t. $f(x+p) = f(x) \; \forall \; x \in \mathbb{R}$.
Recall *Theorem 5.3.9*:

Theorem. Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then the set $f(I) := \{f(x) : x \in I\}$ is a closed bounded interval.

Also recall the *Uniform Continuity Theorem*:

Theorem (Uniform Continuity Theorem). Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I.

We must first show that f is bounded. Let $f : [0,p] \to \mathbb{R}$ be given by $f([0,p]) := f([np,(n+1)p]) \; \forall \; n \in \mathbb{Z}$.

Since $[0,p]$ is a closed bounded interval, we know that by *Theorem 5.3.9*, $f([0,p])$ is also a closed bounded interval. Thus we have that f is bounded.

Now we must show that f is uniformly continuous on $\mathbb{R}$. So, by the *Uniform Continuity Theorem*, since f is continuous on $[0,p]$ and since $[0,p]$ is a closed bounded interval, we have that f is uniformly continuous on $[0,p]$. Since $f([0,p]) = f([np,(n+1)p])$, we have that f is uniformly continuous on $[np,(n+1)p]$ and thus we have that f is uniformly continuous on $\mathbb{R}$.

∎

15. Let f and g be Lipschitz functions on A.

(a) Show that the sum $f+g$ is also a Lipschitz function on A.

Proof. We want to show that $f+g$ is also a Lipschitz function on A.

Recall the definition of a Lipschitz function:

Definition 3. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. If there exists a constant $K > 0$ such that

$$|f(x) - f(u)| \leq K|x - u|$$

for all $x, u \in A$, then f is said to be a **Lipschitz function** (or to satisfy a **Lipschitz condition**) on A.

Since f and g are Lipschitz functions on A, we know that $\forall\ x, y \in A$:

$$\exists\ M_1 > 0 \text{ s.t. } |f(x) - f(y)| < M_1|x - y|$$

$$\exists\ M_2 > 0 \text{ s.t. } |g(x) - g(y)| < M_2|x - y|$$

In order to show that $f + g$ is also a Lipschitz function, we have the following:

$$\begin{aligned}
|(f+g)(x) - (f+g)(y)| &= |f(x) + g(x) - (f(y) + g(y))| \\
&= |f(x) + g(x) - f(y) - g(y)| \\
&= |f(x) - f(y) + g(x) - g(y)| \\
&\leq |f(x) - f(y)| + |g(x) - g(y)| \\
&\leq M_1|x - y| + M_2|x - y| \\
&= (M_1 + M_2)|x - y|
\end{aligned}$$

Thus, we have that $|(f+g)(x) - (f+g)(y)| \leq (M_1 + M_2)|x - y|$. There by the definition of a Lipschitz function, we have that if f and g are Lipschitz, then $f + g$ is also Lipschitz.

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(b) Show that if f and g are bounded on A, then the product fg is a Lipschitz function on A.

Proof. Let f and g be bounded on A. We want to show that the product fg is a Lipschitz function on A.

Recall the definition of a bounded function:

Definition 4. A function $f : A \to \mathbb{R}$ is said to be **bounded on** A if there exists a constant $M > 0$ such that $|f(x)| \leq M$ for all $x \in A$.

So since f and g are bounded on A, we know

$$\exists\ M_1 > 0 \text{ s.t. } |f(x)| \leq M_1\ \forall\ x \in A$$

$$\exists\ M_2 > 0 \text{ s.t. } |g(x)| \leq M_2\ \forall\ x \in A$$

And since $f(x)$ and $g(x)$ are Lipschitz, we know

$$\exists\ K_1 \text{ s.t. } |f(x) - f(u)| \leq K_1|x - u|$$

$$\exists\ K_2 \text{ s.t. } |g(x) - g(u) \leq K_2|x - u|$$

We must show that fg is Lipschitz. Let $x, y \in A$. Then

$$\begin{aligned}
|(fg)(x) - (fg)(y)| &= |f(x)g(x) - f(y)g(y)| \\
&= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\
&= |f(x)(g(x) - g(y)) + g(y)(f(x) - f(y))| \\
&\leq |f(x)(g(x) - g(y))| + |g(y)(f(x) - f(y))| \\
&= |f(x)| \cdot |g(x) - g(y)| + |g(y)| \cdot |f(x) - f(y)| \\
&< M_1 \cdot |g(x) - g(y)| + M_2 \cdot |f(x) - f(y)| \\
&< M_1 K_1 |x - y| + M_2 K_2 |x - y| \\
&= (M_1 K_2 + M_2 K_1)|x - y|
\end{aligned}$$

Thus we have that $|(fg)(x) - (fg)(y)| \leq (M_1 K_1 + M_2 K_2)|x - y| \ \forall \ x, y \in A$. Therefore by the definition of a Lipschitz function, we have that fg is Lipschitz.

$\blacksquare$

(c) Give an example of a Lipschitz function f on $[0, \infty)$ such that its square f^2 is *not* a Lipschitz function.

Consider the function $f(x) := x$, where $x \geq 0$. Then we have that $\forall \ x, y \geq 0$:

$$|f(x) - f(y)| = |x - y| \leq 2|x - y|$$

Thus we have that $f(x)$ is Lipschitz on the interval $[0, \infty)$. However, note that f^2 is not Lipschitz since f^2 is unbounded. Thus f^2 cannot be a Lipschitz function.

2. Section 6.1

1. Use the definition to find the derivative of each of the following functions:

(a) $f(x) := x^3$ for $x \in \mathbb{R}$.

Recall the definition of the derivative:

Definition 5. Let $I \subseteq \mathbb{R}$ be an interval, let $f : I \to \mathbb{R}$, and let $c \in I$. We say that a real number L is the **derivative of f at c** if given any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $x \in I$ satisfies $0 < |x - c| < \delta(\varepsilon)$, then

$$\left| \frac{f(x) - f(c)}{x - c} - L \right| < \varepsilon.$$

In this case we say that f is **differentiable** at c, and we write $f'(c)$ for L. In other words, the derivative of f at c is given by the limit

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

provided this limit exists. (We allow the possibility that c may be the endpoint of the interval.)

Let $h := x - c$. Then $x = c + h$:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{h^3 + 3h^2x + 3hx^2}{h} \\
&= \lim_{h\to 0} \frac{h(h^2 + 3hx + 3x^2)}{h} \\
&= \lim_{h\to 0} (h^2 + 3hx + 3x^2) \\
&= 0^2 + 3 \cdot 0 \cdot x + 3x^2 \\
&= 0 + 0 + 3x^2 \\
&= 3x^2
\end{aligned}$$

Thus $f'(x) = 3x^2$.

(c) $h(x) := \sqrt{x}$ for $x > 0$.

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h\to 0} \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{1}{\sqrt{x+0} + \sqrt{x}} \\
&= \frac{1}{\sqrt{x} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}
\end{aligned}$$

Thus $h'(x) = \frac{1}{2\sqrt{x}}$.

2. Show that $f(x) := x^{1/3},\ x \in \mathbb{R}$ is not differentiable at $x = 0$.

Proof. We must show that $f(x)$ is not differentiable at $x = 0$. By the definition of the derivative, the function $f(x)$ is differentiable at $x = 0$ given that the limit exists.

So, let's find the derivative at $x = 0$:

$$\begin{aligned}
\lim_{x\to c} \frac{f(x)-f(c)}{x-c} &= \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} \\
&= \lim_{x\to 0} \frac{x^{\frac{1}{3}} - 0^{\frac{1}{3}}}{x} \\
&= \lim_{x\to 0} \frac{x^{\frac{1}{3}}}{x} \\
&= \lim_{x\to 0} x^{\frac{1}{3}-1} \\
&= \lim_{x\to 0} x^{-\frac{2}{3}} \\
&= \lim_{x\to 0} \frac{1}{x^{\frac{2}{3}}} \\
&= \frac{1}{0^{\frac{2}{3}}} \\
&= \frac{1}{0} \\
&= \text{undefined}
\end{aligned}$$

Since the limit is undefined when $x = 0$, we have that the limit does not exist at $x = 0$, and thus $f(x)$ is not differentiable at $x = 0$.

■

8. (a) Determine where $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) := |x| + |x+1|$ is differentiable and find the derivative.

We can redefine $f(x)$ as a piecewise function:

$$f(x) := \begin{cases} 2x+1, & x \geq 0 \\ 1, & -1 \leq x < 0 \\ -2x-1, & x < -1 \end{cases}$$

Thus to find where $f(x)$ is differentiable, we will find the derivatives of each of the three functions defined above in the piecewise definition of $f(x)$.

For $x \geq 0$, we have

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{2(x+h)+1-(2x+1)}{h} \\
&= \lim_{h\to 0} \frac{2x+2h-2x}{h} \\
&= \lim_{h\to 0} \frac{2h}{h} \\
&= \lim_{h\to 0} 2 \\
&= 2
\end{aligned}$$

Thus $f'(x) = 2$ when $x \geq 0$.

For $-1 \leq x < 0$:

$$\begin{aligned}
\lim_{x \to c} \frac{f(x) - f(c)}{x - c} &= \lim_{x \to c} \frac{1 - 1}{x - c} \\
&= \lim_{x \to c} \frac{0}{x - c} \\
&= \lim_{x \to c} 0 \\
&= 0
\end{aligned}$$

Thus $f'(x) = 0$ when $-1 \leq x < 0$.

For $x < -1$ we have:

$$\begin{aligned}
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{(-2(x+h) - 1) - (-2x - 1)}{h} \\
&= \lim_{h \to 0} \frac{-2x - 2h - 1 + 2x + 1}{h} \\
&= \lim_{h \to 0} \frac{-2h}{h} \\
&= \lim_{h \to 0} -2 \\
&= -2
\end{aligned}$$

Thus $f'(x) = -2$ when $x < -1$.

Now, we must check for differentiability when $x = -1$ and when $x = 0$.

So when $x = -1$:

$$\begin{aligned}
\lim_{x \to c} \frac{f(x) - f(c)}{x - c} &= \lim_{x \to -1^+} \frac{f(x) - f(-1)}{x + 1} \\
&= \lim_{x \to -1^+} \frac{1 - 1}{x + 1} \\
&= \lim_{x \to -1^+} \frac{0}{x + 1} \\
&= \lim_{x \to -1^+} 0 \\
&= 0
\end{aligned}$$

And

$$\begin{aligned}
\lim_{x\to c}\frac{f(x)-f(c)}{x-c} &= \lim_{x\to -1^-}\frac{f(x)-f(-1)}{x+1} \\
&= \lim_{x\to -1^-}\frac{(-2x-1)-(2\cdot(-1)-1)}{x+1} \\
&= \lim_{x\to -1^-}\frac{-2x-1+3}{x+1} \\
&= \lim_{x\to -1^-}\frac{-2x+2}{x+1} \\
&= -2
\end{aligned}$$

So we have that $f'(-1)$ does not exist since

$$\lim_{x\to -1^-}\frac{f(x)-f(-1)}{x+1} = 0 \neq -2 = \lim_{x\to -1^+}\frac{f(x)-f(-1)}{x+1}$$

Now for when $x=0$:

$$\begin{aligned}
\lim_{x\to c}\frac{f(x)-f(c)}{x-c} &= \lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} \\
&= \lim_{x\to 0^+}\frac{(2x+1)-(2\cdot 0+1)}{x} \\
&= \lim_{x\to 0^+}\frac{2x}{x} \\
&= \lim_{x\to 0^+} 2 \\
&= 2
\end{aligned}$$

And

$$\begin{aligned}
\lim_{x\to c}\frac{f(x)-f(c)}{x-c} &= \lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} \\
&= \lim_{x\to 0^-}\frac{1-1}{x} \\
&= \lim_{x\to 0^-}\frac{0}{x} \\
&= \lim_{x\to 0^-} 0 \\
&= 0
\end{aligned}$$

So we have that $f'(0)$ does not exist since

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = 0 \neq -2 = \lim_{x\to 0^+}\frac{f(x)-f(0)}{x}$$

Thus we have that the function is not differentiable at $x=0$ or at $x=-1$; That is,

$$f'(x) := \begin{cases} 2, & x<0 \\ 0, & -1<x<0 \\ -2, & x<-1 \\ \text{DNE} & x=1 \text{ or } x=0 \end{cases}$$

9. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is an **even function** [that is, $f(-x) = f(x)$ for all $x \in \mathbb{R}$] and has a derivative at every point, then the derivative f' is an **odd function** [that is, $f'(-x) = -f'(x)$ for all $x \in \mathbb{R}$]. Also prove that if $g : \mathbb{R} \to \mathbb{R}$ is a differentiable odd function, then g' is an even function.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be defined such that $f(-x) = f(x)\ \forall\ x \in \mathbb{R}$. Then we have

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}$$

$$f'(-c) = \lim_{x\to -c} \frac{f(x) - f(-c)}{x - (-c)}$$

Now if we change every x for $-x$, then if $-x \to -c$ then $x \to c$. So

$$\begin{aligned}
&= \lim_{x\to c} \frac{f(-x) - f(c)}{-x + c} \\
&= \lim_{x\to c} \frac{f(x) - f(c)}{-(x - c)} && \because f(-x) = f(x)\ \forall\ x \in \mathbb{R} \\
&= -\lim_{x\to c} \frac{f(x) - f(c)}{x - c} \\
&= -f'(c)
\end{aligned}$$

Thus we have that $f'(-c) = -f'(c)$, an odd function.

$\therefore$ If f is an even function, then f' is an odd function. ■

Proof. Let $g : \mathbb{R} \to \mathbb{R}$ be defined such that $g(-x) = -g(x)\ \forall\ x \in \mathbb{R}$. Then

$$g'(c) = \lim_{x\to c} \frac{g(x) - g(c)}{x - c}$$

And

$$g'(-c) = \lim_{x\to -c} \frac{g(x) - g(-c)}{x - (-c)}$$

Let us change every x for $-x$ if $-x \to -c$ then $x \to c$. So

$$\begin{aligned}
&= \lim_{x\to c} \frac{g(-x) - (-g(c))}{-x + c} \\
&= \lim_{x\to c} \frac{-g(x) + g(c)}{-x + c} && \because g(-x) = -g(x)\ \forall\ x \in \mathbb{R} \\
&= \lim_{x\to c} \frac{-(g(x) - g(c))}{-(x - c)} \\
&= \lim_{x\to c} \frac{g(x) - g(c)}{x - c} \\
&= g'(c)
\end{aligned}$$

Thus we have that $g'(-c) = g'(c)$, an even function.

$\therefore$ If g is an odd function, then g' is an even function. ■

11. Assume that there exists a function $L : (0, \infty) \to \mathbb{R}$ such that $L'(x) = 1/x$ for $x > 0$. Calculate the derivatives of the following functions:

(a) $f(x) := L(2x + 3)$ for $x > 0$

Recall the *Chain Rule*:

Theorem (Chain Rule). Let I, J be intervals in $\mathbb{R}$, let $g : I \to \mathbb{R}$ and $f : J \to \mathbb{R}$ be functions such that $f(J) \subseteq I$, and let $c \in J$. If f is differentiable at c and if g is differentiable at $f(c)$, then the composite function $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c).$$

So utilizing the *Chain Rule*:

$$\begin{aligned} f'(x) &= (L(2x+3))' \\ &= \frac{1}{2x+3} \cdot (2x+3)' \\ &= \frac{2}{2x+3} \end{aligned}$$

So $f'(x) = \frac{2}{2x+3}$.

(c) $h(x) := L(ax)$ for $a > 0, x > 0$

Once again utilizing the *Chain Rule*:

$$\begin{aligned} h'(x) &= (L(ax))' \\ &= \frac{1}{ax} \cdot (ax)' \\ &= \frac{1}{\cancel{a}x} \cdot \cancel{a} \\ &= \frac{1}{x} \end{aligned}$$

So $h'(x) = \frac{1}{x}$.

3. **(a)** Show that $e^x = 2\cos x + 1$ for some $x \in [0, \pi]$

Proof. Let $h(x) := e^x - 2\cos(x) - 1$. Then we have that $h(0) = 2 < 0$ and $h(\pi) = e^\pi + 1 > 0$. Since h is continuous, by the *Intermediate Value Theorem*, we know that there exists $c \in (0, \pi)$ s.t. $h(c) = 0$. Thus we have that $e^c = 2\cos(c) + 1$. ■

(b) Let $h(x) = x^3 + 2x + 1$. Compute $h(1)$, $h'(1)$ and $[h^{-1}]'(1)$.

$$h(1) = 1^3 + 2(1) + 1 = 1 + 2 + 1 = 4$$

So $h(1) = 4$.

Now, as for $h'(1)$:

$$\begin{aligned}
h'(1) &= \lim_{x\to 1} \frac{f(x) - f(1)}{x-1} \\
&= \lim_{x\to 1} \frac{x^3 + 2x + 1 - 4}{x-1} \\
&= \lim_{x\to 1} \frac{x^3 + 2x - 3}{x-1} \\
&= \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2 + x + 3)}{\cancel{x-1}} \\
&= \lim_{x\to 1} (x^2 + x + 3) \\
&= 1^2 + 1 + 3 \\
&= 1 + 1 + 3 \\
&= 5
\end{aligned}$$

So $h'(1) = 5$.

As for $[h^{-1}]'(1)$:

We first need to find h^{-1}. Since $h(1) = 4$, we must solve $x^3 + 2x + 1 = 1$. So

$$\begin{aligned}
x^3 + 2x + 1 &= 1 \\
x^3 + 2x &= 0 \\
x(x^2 + 1) &= 0 \\
x(x+1)(x-1) &= 0
\end{aligned}$$

Thus $x = 0$.

Since $h'(1)$ exists and since $h'(1) \neq 0$, we have that

$$[h^{-1}]'(1) = \frac{1}{h'(1)} = \frac{1}{5}$$

(c) Show $f(x) = x^2$ for $x \in (-2, 1]$ is a Lipschitz function.

Proof. To show that $f(x)$ is Lipschitz, we have the following:

$$\begin{aligned}\left|\frac{f(x)-f(u)}{x-u}\right| &= \left|\frac{x^2-u^2}{x-u}\right| \\ &= \left|\frac{\cancel{(x-u)}(x+u)}{\cancel{x-u}}\right| \\ &= |(x+u)| \\ &\leq 8\end{aligned}$$

■

(d) Suppose that $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ are continuous with $f(a)\leq g(a)$ and $f(b)\geq g(b)$. Prove that $f(c)=g(c)$ for some $c\in[a,b]$.

Proof. Let $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ be continuous functions such that $f(a)\leq g(a)$, and $f(b)\geq g(b)$. We want to show that $f(c)=g(c)$ for some $c\in[a,b]$.

Let $h(x):=f(x)-g(x)$. Since both f and g are continuous, we know that h is also continuous. And since h is continuous, we know that $h(a)<0$, and $h(b)>0$.

Recall *Bolzano's Intermediate Value Theorem*:

Theorem (Bolzano's Intermediate Value Theorem). Let I be an interval and let $f:I\to\mathbb{R}$ be continuous on I. If $a,b\in I$ and if $k\in\mathbb{R}$ satisfies $f(a)<k<f(b)$, then there exists a point $c\in I$ between a and b such that $f(c)=k$.

Thus by *Bolzano's Intermediate Value Theorem*, we have that $\exists\, c\in(a,b)$ s.t. $h(c)=0$. Thus we have that $h(c)=0=f(c)-g(c) \implies g(c)=f(c)$. Thus $g(c)=f(c)$.

■

(e) Prove: If f is uniformly continuous on (a,b), then for any sequence x_n in (a,b) that converges, then the sequence $(f(x_n))$ is Cauchy.

Proof. Let f be uniformly continuous on (a,b), and let (x_n) be a sequence in (a,b) such that (x_n) converges.

Recall the *Cauchy Convergence Criterion*:

Theorem. A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Since we have that (x_n) is a convergent sequence, we know that (x_n) is a Cauchy sequence by the *Cauchy Convergence Criterion.*

Also recall *Theorem 5.4.7*:

Theorem. If $f : A \to \mathbb{R}$ is uniformly continuous on a subset A of $\mathbb{R}$ and if (x_n) is a Cauchy sequence in A, then $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$.

Since (x_n) is a sequence in (a, b) and since f is uniformly continuous on (a, b), we have that by *Theorem 5.4.7*, the sequence $(f(x_n))$ is Cauchy.

■

(f) Prove: If $f : [a, b] \to [a, b]$ is a contraction, then f has a unique fixed point c satisfying $f(c) = c$ for some $c \in [a, b]$.

Proof. Since f is a contraction, we know that f is Lipschitz with $k \in (0, 1)$. This in turn implies that f is continuous.

By the *Brower Fixed Point Theorem*, there exists c s.t. $f(c) = c$. We want to show the uniqueness of this point.

Assume that there exists c_1, c_2 s.t. $f(c_1) = c_1$ and $f(c_2) = c_2$, and $c_1 \neq c_2$.

Assume without loss of generality that $c_1 < c_2$.

Consider the interval $[c_1, c_2] \leq [a, b]$. Since f is a contraction, we have that

$$|f(c_1) - f(c_2)| \leq K|c_1 - c_2| < |c_1 - c_2|$$

Thus we have that $|c_1 - c_2| \leq |c_1 - c_2|$, which is a contradiction. Thus if f is a contraction, then f has a unique fixed point c satisfying $f(c) = c$.

■

4. Prove or justify, if true; Provide a counterexample, if false. For all parts, assume f is a function defined on the given interval or set.

(a) If f is bounded and continuous on A, then f is uniformly continuous on A.

This is a false statement. Consider $f : (0, 2) \to \mathbb{R}$ given by $f(x) := \frac{1}{x}$. It is clear that f is continuous in $(0, 2)$ as it is the quotient of two polynomials and the denominator is never equal to 0. Let $x, u \in (0,2)$. First, suppose $x > \frac{u}{2}$. So,

$$\begin{aligned}
|f(x) - f(u)| &= \left|\frac{1}{x} - \frac{1}{u}\right| \\
&= \left|\frac{x-u}{xu}\right| \\
&= \frac{|x-u|}{xu} && \because\ x > 0,\ u > 0,\ \forall\ x, u \in (0,2) \\
&< \frac{2|x-u|}{u \cdot u} && \because\ x > \frac{u}{2} \implies \frac{1}{x} < \frac{2}{u} \\
&= \frac{2|x-u|}{u^2} \\
&= \frac{2}{u^2}|x-u| \\
&< \varepsilon
\end{aligned}$$

So $|x-u| < \frac{u^2\varepsilon}{2}$. Thus, let $\delta = \min\{\frac{u^2\varepsilon}{2}, \frac{u}{2}\}$. Hence $f(x)$ is continuous on $(0,2)$.

Recall that in order for $f(x)$ to be considered uniformly continuous, $\forall\ \delta > 0$ must always satisfy $|x-u| < \delta\ \forall\ \varepsilon > 0$. However, if we let $\varepsilon = 1$, then we have that $\forall\ \delta > 0$:

$$x := \min\{\delta, 1\},\ u := \frac{x}{2} \implies |x-u| = \frac{x}{2} < \delta$$

but

$$\left|\frac{1}{x} - \frac{1}{u}\right| = \left|\frac{1}{x} - \frac{2}{x}\right| = \left|\frac{1}{x}\right| \geq 1 = \varepsilon$$

Thus $f(x)$ is both bounded and continuous on $A := (0,2)$, but $f(x)$ is not uniformly continuous.

(b) If f is uniformly continuous on A, then f is bounded on A.

This is a false statement. Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) := 3x + 7$. Then if we let $x, u \in \mathbb{R}$:

$$\begin{aligned}
|3x + 7 - (3u + 7)| &= |3x + 7 - 3u - 7| \\
&= |3x - 3u| \\
&= 3|x-u| \\
&< \varepsilon
\end{aligned}$$

So, let $\delta := \frac{\varepsilon}{3}$. Thus we have that $f(x)$ is uniformly continuous. However, since $\mathbb{R}$ is unbounded, we have that f is not bounded on $A := \mathbb{R}$ since $\nexists\ M > 0$ s.t. $|f(x)| \leq M\ \forall\ x \in \mathbb{R}$.

(c) If f is uniformly continuous on (a, b), then f is bounded on (a, b).

This is a true statement. We prove it by way of contradiction:

Proof. Assume that f is uniformly continuous on (a, b), and suppose by way of contradiction that f is not bounded on (a, b). Then we have that $\forall\, n \in \mathbb{N}$, there exists a corresponding $f(x_n)$ s.t. $|f(x_n)| > n$, where $x_n \in (a, b)$. By the *Bolzano-Weirstrass Theorem*, there exists a convergent subsequence $(x_{n_k}) \subseteq (x_n)$.

Recall *Lemma 3.5.2*:

Lemma. If $X = (x_n)$ is a convergent sequence of real numbers, then X is a Cauchy sequence.

Thus by *Lemma 3.5.2*, (x_{n_k}) is a Cauchy sequence.

Since f is uniformly continuous on (a, b), we know that $f(x_{n_k})$ is also Cauchy. However, this is a contradiction since $f(x_{n_k})$ is clearly divergent. Thus, we have that f is bounded on (a, b).

■

(d) If f is bounded on A, then f is uniformly continuous on A.

This is a false statement. Consider $f : (0, 2) \to \mathbb{R}$ given by $f(x) := \frac{1}{x}$. It is clear that f is continuous in $(0, 2)$ as it is the quotient of two polynomials and the denominator is never equal to 0. Let $x, u \in (0, 2)$. First, suppose $x > \frac{u}{2}$. So,

$$\begin{aligned}
|f(x) - f(u)| &= \left|\frac{1}{x} - \frac{1}{u}\right| \\
&= \left|\frac{x-u}{xu}\right| \\
&= \frac{|x-u|}{xu} && \because\ x > 0,\ u > 0,\ \forall\ x, u \in (0,2) \\
&< \frac{2|x-u|}{u \cdot u} && \because\ x > \frac{u}{2} \implies \frac{1}{x} < \frac{2}{u} \\
&= \frac{2|x-u|}{u^2} \\
&= \frac{2}{u^2}|x-u| \\
&< \varepsilon
\end{aligned}$$

So $|x - u| < \frac{u^2 \varepsilon}{2}$. Thus, let $\delta = \min\{\frac{u^2\varepsilon}{2}, \frac{u}{2}\}$. Hence $f(x)$ is continuous on $(0, 2)$.

Recall that in order for $f(x)$ to be considered uniformly continuous, $\forall\ \delta > 0$ must always satisfy $|x - u| < \delta\ \forall\ \varepsilon > 0$. However, if we let $\varepsilon = 1$, then we have that $\forall\ \delta > 0$:

$$x := \min\{\delta, 1\},\ u := \frac{x}{2} \implies |x - u| = \frac{x}{2} < \delta$$

but

$$\left|\frac{1}{x}-\frac{1}{u}\right| = \left|\frac{1}{x}-\frac{2}{x}\right| = \left|\frac{1}{x}\right| \geq 1 = \varepsilon$$

Thus $f(x)$ is bounded on $A := (0, 2)$, but $f(x)$ is not uniformly continuous on A.

(e) The derivative of f at $x = c$ is defined by $f'(c) = \lim\limits_{x\to c} \frac{f(x)-f(c)}{x-c}$ provided the limits exists.

This is true since this is the definition of the derivative.

(f) If f is continuous at c, then f is differentiable at c.

This is a false statement. Consider $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) := \begin{cases} \frac{1}{q} & x = \frac{p}{q},\ x \in \mathbb{Q},\ p \in \mathbb{Z},\ q \in \mathbb{N} \text{ s.t. } \gcd(p, q) = 1 \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

This is Thomae's function, which we know is continuous in the irrationals, and discontinuous in the rationals. So, we know that $f(c)$ is continuous at $c = \sqrt{2}$, but notice that f is not differentiable at $c = \sqrt{2}$. In fact, f is not differentiable for any values of c. We know that it's not differentiable when $x \in \mathbb{Q}$ since f is not continuous for any value $x \in \mathbb{Q}$. As for our particular case where $c = \sqrt{2}$:

$$\begin{aligned}
\lim_{x\to c} \frac{f(x)-f(c)}{x-c} &= \lim_{x\to\sqrt{2}} \frac{f(x)-f(\sqrt{2})}{x-\sqrt{2}} \\
&\Downarrow \\
x \in \mathbb{Q} &\implies x = \frac{p}{q},\ p \in \mathbb{Z},\ q \in \mathbb{N},\ \gcd(p, q) = 1 \\
&\Downarrow \\
&= \lim_{x\to\sqrt{2}} \frac{\frac{1}{q}-0}{x-0} \\
&= \lim_{x\to\sqrt{2}} \frac{\frac{1}{q}}{x} \\
&= \lim_{x\to\sqrt{2}} \frac{\frac{1}{q}}{\frac{p}{q}} && \because\ x = \frac{p}{q} \\
&= \lim_{x\to\sqrt{2}} \frac{q}{qp} \\
&= \lim_{x\to\sqrt{2}} \frac{1}{p} \\
&= \text{DNE}
\end{aligned}$$

Thus the limit does not exist since it is a contradiction that as $x \to \sqrt{2},\ x \in \mathbb{Q}$, and since $\sqrt{2} \notin \mathbb{Q}$, it's impossible to substitute $\sqrt{2}$ for x. Additionally, since $p \in \mathbb{Z}$, p can equal 0, and thus the function is undefined at this point, again rendering the limit non-existent. Thus f is continuous at $c = \sqrt{2}$, but f is not differentiable at $c = \sqrt{2}$.

(g) If f is differentiable at c, then f is continuous at c.

This is true by *Theorem 6.1.2*:

Theorem. If $f : I \to \mathbb{R}$ has a derivative at $c \in I$, then f is continuous at c.

(h) If f is differentiable on $[a, b]$, then f is uniformly continuous on $[a, b]$.

This is a true statement.

Proof. Let $f : [a, b] \to [a, b]$ be a function such that f is differentiable $\forall\ x \in A$. Let $x, c \in A$, and without loss of generality, let $x \neq c$ (since f is differentiable everywhere, f is also differentiable at $c = 0$.).

Recall *Carathéodory's Theorem*:

Theorem (Carathéodory's Theorem). Let f be defined on an interval I containing the point c. Then f is differentiable at c if and only if there exists a function ϕ on I that is continuous at c and satisfies

$$f(x) - f(c) = \phi(x)(x - c) \quad \text{for} \quad x \in I$$

In this case, we have $\phi(c) = f'(c)$.

Since f is differentiable at all points, we know that $\exists\ \phi$ s.t. $f(x) - f(c) = \phi(x)(x - c)$. Let ϕ be given by $\phi(x) := f'(c)$.

Also, recall the *Mean Value Theorem*:

Theorem (Mean Value Theorem). Suppose that f is continuous on a closed interval $I := [a, b]$, and that f has a derivative in the open interval (a, b). Then there exists at least one point c in (a, b) such that

$$f(b) - f(a) = f'(c)(b - a)$$

Thus we have that $f'(c) = \frac{f(x)-f(c)}{x-c} \leq M$ since $f'(x) \leq M\ \forall\ x \in \mathbb{R}$. Thus $|x - c| < \frac{\varepsilon}{M}|f(x) - f(c)| < \varepsilon$.

$$\begin{aligned} |f(x) - f(c)| &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot (x - c) \\ &= \lim_{x \to c} f(x) - f(c) \\ &< M \cdot \varepsilon \end{aligned}$$

Thus let $\delta = \frac{\varepsilon}{M}$.

Recall the definition of a *Lipschitz Function*:

Definition 6. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. If there exists a constant $K > 0$ such that

$$|f(x) - f(u)| \le K|x - u| \tag{4}$$

for all $x, u \in A$, then f is said to be a **Lipschitz function** (or to satisfy a **Lipschitz condition**) on A.

The condition (4) that a function $f : I \to \mathbb{R}$ on an interval I is a Lipschitz function can be interpreted geometrically as follows. If we write the condition as

$$\left|\frac{f(x) - f(u)}{x - u}\right| \le K,\ x, u \in I,\ x \ne u,$$

then the quantity inside the absolute values is the slope of a line segment joining the points $(x, f(x))$ and $(u, f(u))$. Thus a function f satisfies a Lipschitz condition if and only if the slopes of all line segments joining two points on the graph of $y = f(x)$ over I are bounded by some number K.

If we let $M\delta = K$, we have that f is a Lipschitz function.

Now, recall *Theorem 5.4.3*:

Theorem 0.1. If $f : A \to \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on A.

Also, recall the *Continuous Extension Theorem*:

Theorem (Continuous Extension Theorem). A function f is uniformly continuous on the interval (a, b) if and only if it can be defined at the endpoints a and b such that the extended function is continuous on $[a, b]$.

And recall *Theorem 6.1.1*:

Theorem. If $f : I \to \mathbb{R}$ has a derivative at $c \in I$, then f is continuous at c.

So, by *Theorem 5.4.3* we have that f is uniformly continuous.

By *Theorem 6.1.1*, since f is differentiable $\forall\ c \in [a, b]$, f is continuous on $[a, b]$.

Thus by the *Continuous Extension Theorem*, since f is continuous on $[a, b]$, we have that f is uniformly continuous on (a, b).

Recall the *Uniform Continuity Theorem*:

Theorem (Uniform Continuity Theorem). Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I.

Since $[a, b]$ is a closed interval, and since f is continuous on $[a, b]$, we have that by the *Uniform Continuity Theorem*, f is uniformly continuous on $[a, b]$.

$\therefore$ If f is differentiable on $[a, b]$, then f is uniformly continuous on $[a, b]$.

$\blacksquare$

(i) If f is differentiable on (a, b) and $f(a) = f(b) = 0$, then f is uniformly continuous at $[a, b]$.

This is a false statement. Consider $f : (0, \pi) \to \mathbb{R}$ given by $f(x) := x\sin(x)$. Then we have that

$$f(a) = f(0) = 0\sin(0) = 0 \cdot 0 = 0 = \pi \cdot 0 = \pi\sin(\pi) = f(\pi) = f(b)$$

Recall *Theorem 6.1.2*:

Theorem. Let $I \subseteq \mathbb{R}$ be an interval, let $c \in I$, and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be functions that are differentiable at c. Then:

i. If $\alpha \in \mathbb{R}$, then the function αf is differentiable at c, and

$$(\alpha f)'(c) = \alpha f'(c)$$

ii. The function $f + g$ is differentiable at c, and

$$(f + g)'(c) = f'(c) + g'(c)$$

iii. (Product Rule) The function fg is differentiable at c, and

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

iv. (Quotient Rule) If $g(c) \neq 0$, then the function f/g is differentiable at c, and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$$

So, if we let $g(x) := x$, and $h(x) := \sin(x)$, we then have

$$\begin{aligned}
\lim_{c\to 0}\frac{f(x+c)-f(x)}{c} &= \lim_{c\to 0}\frac{g(x+c)-g(x)}{c}\cdot h(x) + g(x)\cdot\lim_{c\to 0}\frac{h(x+c)-h(x)}{c}\\
&= \lim_{c\to 0}\frac{x+c-x}{c}\cdot\sin(x) + x\cdot\lim_{c\to 0}\frac{\sin(x+c)-\sin(x)}{c}\\
&= \lim_{c\to 0}\frac{c}{c}\cdot\sin(x) + x\cdot\lim_{c\to 0}\frac{(\sin(x)\cos(c)+\sin(c)\cos(x))-\sin(x)}{c}\\
&(\because\ \sin(a+b)=\sin(a)\cos(b)+\sin(b)\cos(a))\\
&= \lim_{c\to 0} 1\cdot\sin(x) + x\cdot\lim_{c\to 0}\left(\cos(x)\cdot\frac{\sin(c)}{c}+\sin(x)\cdot\frac{\cos(c)-1}{c}\right)\\
&= \sin(x) + x\cdot\lim_{c\to 0}\left(\cos(x)\cdot\frac{\sin(c)}{c}+\sin(x)\cdot\frac{\cos^2(c)-1}{(\cos(c)+1)c}\right)\\
&= \sin(x) + x\cdot\lim_{c\to 0}\left(\cos(x)\cdot\frac{\sin(c)}{c}-\sin(x)\cdot\frac{\sin^2(c)}{(\cos(c)+1)c}\right)\\
&(\because\ \sin^2(c)+\cos^2(c)=1)\\
&= \sin(x) + x\cdot\lim_{c\to 0}\left(\left(\cos(x)-\frac{\sin(x)\sin(c)}{\cos(c)+1}\right)\frac{\sin(c)}{c}\right)\\
&= \sin(x) + x\cdot\left(\lim_{c\to 0}\left(\cos(x)-\frac{\sin(x)\sin(c)}{\cos(c)+1}\right)\right)\cdot\left(\lim_{c\to 0}\frac{\sin(c)}{c}\right)\\
&= \sin(x) + x\cdot\cos(x)\left(\lim_{c\to 0}\frac{\sin(c)}{c}\right)\\
&(\because\ \lim_{c\to 0}\left(\cos(x)-\frac{\sin(x)\sin(c)}{\cos(c)+1}\right) = \cos(x)-\frac{\sin(x)\sin(0)}{\cos(0)+1} = \cos(x)\\
&\text{by continuity})\\
&= \sin(x) + x\cos(x)\\
&(\because\ \lim_{c\to 0}\frac{\sin(c)}{c} = 1)
\end{aligned}$$

Thus $x\sin(x)$ is differentiable on $(0,\pi)$.

Now, let $(x_n),(u_n)\subseteq(0,\pi)$ be given by $x_n := \pi$, and $u_n := \pi + \frac{1}{n}$ for $n\in\mathbb{N}$. Then we have

$$(x_n-u_n) = \pi - (\pi + \frac{1}{n}) = -\frac{1}{n}$$

and thus

$$\lim_{n\to\infty}(x_n-u_n) = \lim_{n\to\infty} -\frac{1}{n} = 0$$

Now we have

$$\begin{aligned}
|f(x_n) - f(u_n)| &= \left|\pi\sin(\pi) - \left(\pi + \frac{1}{n}\right)\sin(\pi + \frac{1}{n})\right| \\
&= \left|\pi \cdot 0 - \left(\pi + \frac{1}{n}\right)\sin\left(\pi + \frac{1}{n}\right)\right| \\
&= \left|-\left(\pi + \frac{1}{n}\right)\sin\left(\pi + \frac{1}{n}\right)\right| \\
&= \left(\pi + \frac{1}{n}\right)\sin\left(\pi + \frac{1}{n}\right) \\
&= \left(\pi + \frac{1}{n}\right)\sin(\frac{1}{n}) && \because \sin(\pi) = 0 \\
&= \pi\sin\left(\frac{1}{n}\right) + \left(\frac{1}{n}\right)\sin\left(\frac{1}{n}\right)
\end{aligned}$$

We notice that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ and $(\sin\left(\frac{1}{n}\right))$ is a bounded sequence in $(0, \pi)$. Thus

$$\lim_{n\to\infty}\left(\frac{1}{n}\sin\frac{1}{n}\right) = 0$$

and

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1 \; \forall \; x \in (0, \pi)$$

This yields

$$\lim_{n\to\infty}\pi\sin\left(\frac{1}{n}\right) = \pi \lim_{\frac{1}{n}\to 0}\left(\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}\right) = \pi$$

Thus $\lim\limits_{n\to\infty}(f(x_n) - f(u_n)) = \pi$.

Let $\varepsilon = \pi$. Then $\exists \; k \in \mathbb{N}$ s.t. $\forall \; n \geq k$:

$$\begin{aligned}
\pi - \varepsilon < f(x_n) - f(u_n) &< \pi + \varepsilon \\
f(x_n) - f(u_n) &> \pi \\
|f(x_n) - f(u_n)| &> \pi \\
|f(x_{n+k}) - f(u_{n+k})| &> \pi
\end{aligned}$$

Thus, for $\varepsilon_0 = \pi$, and the two sequences (x_{n+k}), (u_{n+k}), by the *Nonuniform Continuity Criteria*, f is not uniformly continuous on $(0, \pi)$

$\therefore$ If f is differentiable on (a, b), and $f(a) = f(b) = 0$, then f is **not** uniformly continuous on $[a, b]$.