Real Analysis Homework 5

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September 30, 2018

1. For the following sequences, i) write out the first 5 terms, ii) Use the Monotone Sequence Property to show that the sequences converges.

(a) Section 3.3

2) Let $x_1 > 1$ and $x_{n+1} := 2 - 1/x_n$ for $n \in \mathbb{N}$. Show that (x_n) is bounded and monotone. Find the limit.

The first five terms of this sequence are $x_1 \geq 2, x_2 \geq \frac{3}{2}, x_3 \geq \frac{4}{3}, x_4 \geq \frac{5}{4}, x_5 \geq \frac{6}{5}, \cdots \approx x_1 \geq 2, x_2 \geq 1.5, x_3 \geq 1.3333, x_4 \geq 1.25, x_5 \geq 1.2, \ldots$. This sequence appears to be decreasing.

Recall the Monotone Sequence Property:

Theorem. Monotone Sequence Property A monotone sequence of real numbers is convergent if and only if it is bounded. Further,

A. If $X = (x_n)$ is a bounded increasing sequence, then

$$\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\}$$

B. If $Y = (y_n)$ is a bounded decreasing sequence, then

$$\lim(y_n) = \inf\{y_n : n \in \mathbb{N}\}$$

To show that this sequence converges, we must first find the possible limit points (fixed points) of this sequence. So,

$$\begin{aligned} x &= 2 - \frac{1}{x} \\ x^2 &= 2x - 1 \\ x^2 - 2x + 1 &= 0 \\ (x-1)^2 &= 0 \end{aligned}$$

Thus, $x = 1$ is a possible limit of this sequence.

Now, we will prove that (x_n) is bounded by 1, and since we hypothesized that (x_n) is decreasing, we say that (x_n) is bounded below by 1.

Proof. We want to show that the sequence (x_n) is bounded below by 1; that is, we want to show that $1 \leq x_n,\ \forall\ n \in \mathbb{N}$. We prove it by method of mathematical induction.

Basis Step: Let $n = 1$. Then

$$\begin{aligned} x_n &\geq x_{n+1}, && \text{by the definition of decreasing,} \\ x_1 &\geq x_{1+1} \\ x_1 &\geq x_2 \end{aligned}$$

Since $x_1 > 1 \Rightarrow \frac{1}{x_1} < 1$, we have

$$x_2 = 2 - \frac{1}{x_1} > 1$$

$$\Rightarrow 1 < x_2 < 2.$$

Since $x_1 > 1$ and because $1 < x_2 < 2$, we have that $x_1 \geq x_2$.

Inductive Step: Assume $1 < x_n < 2,\ \forall\ n \in \mathbb{N}$.

Show: Now we want to show that $x_n \leq x_{n+1}$.
So,

$$\begin{gathered} 1 < x_n < 2 \\ 1 > \frac{1}{x_n} > \frac{1}{2} \\ -1 < -\frac{1}{x_n} < -\frac{1}{2} \\ 1 < 2 - \frac{1}{x_n} < 2 - \frac{1}{2} < 2 \\ 1 < x_{n+1} < 2 \end{gathered}$$

Thus we have that (x_n) is bounded between 1 and 2. ■

Now we need to show that (x_n) is monotone decreasing; that is, we must show that $x_1 \geq x_2 \geq \cdots \geq x_n$.

Proof. We want to show that $x_1 \geq x_2 \geq \cdots \geq x_n,\ \forall\ n \in \mathbb{N}$. We prove it by method of mathematical induction.

Basis Step: Let $n = 1$. Then since $x_1 > 1$ is given, we have that $\frac{1}{x_1} < 1$. This yields $x_2 = 2 - \frac{1}{x_1} > 1$, as was determined for the boundedness proof, and thus we have that $1 < x_2 < 2$. This means that $1 > \frac{1}{x_2} > \frac{1}{2}$, and since $\frac{1}{2} \leq \frac{1}{x_n}$, we have $x_2 \geq x_1$.

Inductive Step: Assume $x_n \geq x_{n+1}\ \forall\ n \in \mathbb{N}$.

Show: We now want to show that $x_{n+2} \leq x_{n+1}$.
So,

$$x_{n+2} = 2 - \frac{1}{x_{x+1}}$$

Recall the inductive hypothesis, in that $x_n \geq x_{n+1} \Rightarrow \frac{1}{x_n} \leq \frac{1}{x_{n+1}}$. Thus,

$$-\frac{1}{x_n} \geq -\frac{1}{x_{n+1}}$$

$$\Rightarrow 2 - \frac{1}{x_n} \leq 2 - \frac{1}{x_{n+1}}$$

$$x_{n+1} \leq x_{n+2}$$

$\therefore$ we have that $x_1 \geq x_2 \geq \cdots \geq x_n, \ \forall\, n \in \mathbb{N}$. ■

Thus (x_n) is monotone decreasing.

By the *Monotone Sequence Property*, since we have shown that (x_n) is both bounded (and thus converges), and that (x_n) is monotone decreasing, we have that

$$\begin{aligned} \lim(x_n) &= \inf\{x_n : n \in \mathbb{N}\} \\ &= \inf(1, 2) \\ &= 1 \end{aligned}$$

Hence the sequence converges to the previously found possible limit of 1.

3) Let $x_1 > 1$ and $x_{n+1} := 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$. Show that (x_n) is decreasing and bounded below by 2. Find the limit.

The first 5 terms of this sequence are $x_1 \geq 2, x_2 \geq 2, x_3 \geq 2, x_4 \geq 2, x_5 \geq 2, \ldots$.
Notice the following, however:

$$\begin{aligned} x_{n+1} \leq x_n &\iff 1 + \sqrt{x_n - 1} \leq x_n \\ &\iff \sqrt{x_n - 1} \leq x_n - 1 \end{aligned}$$

which we know is always true since the square root function is a decreasing function.

Now we must find the possible limit points (fixed points) of this sequence. So,

$$\begin{aligned}
x &= 1 + \sqrt{x-1} \\
x - 1 &= \sqrt{x-1} \\
x - 1 &= (x-1)^2 \\
x - 1 &= x^2 - 2x + 1 \\
(x-1) - (x^2 - 2x + 1) &= 0 \\
-x^2 + 3x - 2 &= 0 \\
-(x^2 - 3x + 2) &= 0 \\
-(x-1)(x-2) &= 0 \\
(x-1)(x-2) &= 0
\end{aligned}$$

Thus $x = 1$, or $x = 2$. These are the possible limits of (x_n). Since we hypothesized that (x_n) is decreasing, then we say that (x_n) is bounded below by 2, since we are given that $x_1 > 1$.

Now we will prove that (x_n) is bounded below by 2.

Proof. We want to show that (x_n) is bounded below by 1; that is, we want to show that $1 \leq x_n$, $\forall\ n \in \mathbb{N}$. We prove it by method of mathematical induction.

Basis Step: Let $n = 1$. Then we are given that $x_1 \geq 2$.

Inductive Step: Assume that $x_n \geq 2$, $\forall\ n \in \mathbb{N}$.

Show: We now want to show that $x_{n+1} \geq 2$, $\forall\ n \in \mathbb{N}$.

So,

$$\begin{aligned}
x_{n+1} &= 1 + \sqrt{x_n - 1} \\
&\geq 1 + \sqrt{2-1} \\
&= 1 + 1 \\
&= 2
\end{aligned}$$

Thus, $x_n \geq 2$, $\forall\ n \in \mathbb{N}$. By the definition of boundedness, we have that (x_n) is bounded below by 2. ■

Since we have also shown earlier that (x_n) is monotone decreasing, we have that by the monotone sequence property, since (x_n) is bounded, (x_n) converges, and since (x_n) is monotone decreasing, we have:

$$\begin{aligned}
\lim(x_n) &= \inf\{x_n : n \in \mathbb{N}\} \\
&= 2
\end{aligned}$$

7) Let $x_1 := a > 0$ and $x_{n+1} := x_n + 1/x_n$ for $n \in \mathbb{N}$. Determine whether (x_n) converges or diverges.

The first 5 terms of this sequence are $x_1 \geq 1, x_2 \geq 2, x_3 \geq \frac{5}{2}, x_4 \geq \frac{29}{10}, x_5 \geq \frac{941}{290}, \cdots \approx x_1 \geq 1, x_2 \geq 2, x_3 \geq 2.5, x_4 \geq 2.9, x_5 \geq 3.244828, \ldots$. This sequence appears to be increasing. We show this to be true as follows:

$$\begin{aligned} x_{n+1} \geq x_n &\iff x_n + \frac{1}{x_n} \geq x_n \\ &\iff x_n^2 + 1 \geq x_n^2 \\ &\iff 1 \geq 0 \end{aligned}$$

which is true. However, notice that one of the terms of the sequence is x_n. We know that x_n is an unbounded sequence. Thus, we can infer that (x_n) is unbounded above. We show this as follows:

$$\begin{aligned} x_{n+1}^2 &= \left(x_n + \frac{1}{x_n}\right)^2 \\ &= x_n^2 + 2 + \frac{1}{x_n^2} \\ &> x_n^2 + 2 \end{aligned}$$

Since:

$$x_{n+1}^2 > x_n^2 + 2 > x_{n-1}^2 + 4 > \cdots > x_1^2 + 2 \cdot n = a^2 + 2 \cdot n$$
$$\Downarrow$$
$$x_n > \sqrt{a^2 + 2 \cdot (n-1)}$$

Since the right hand side of this inequality is unbounded, the left hand side is also unbounded.

Thus we have that this sequence (x_n) is unbounded above.

Since this sequence is increasing and unbounded above, we have that the sequence is divergent.

8) Let (a_n) be an increasing sequence, (b_n) be a decreasing sequence, and assume that $a_n \leq b_n$ for all $n \in \mathbb{N}$. Show that $\lim(a_n) \leq \lim(b_n)$, and thereby deduce the Nested Intervals Property 2.5.2 from the Monotone Convergence Theorem 3.3.2.

Since (a_n) is an increasing sequence, we know that $(a_1 \leq a_2 \leq \cdots \leq a_n)$, and since (b_n) is a decreasing sequence, we know that $(b_1 \geq b_2 \geq \cdots \geq b_n)$. Also, since we have that $a_n \leq b_n$, $\forall\ n \in \mathbb{N}$, we know that (a_n) is bounded above by (b_1). Thus, by the *Monotone Convergence Theorem*, we know that

$$\lim(a_n) = \sup\{a_n : n \in \mathbb{N}\}$$

Also, since (b_n) is a decreasing sequence such that it is bounded below by (a_1), by the *Monotone Convergence Theorem*, we have

$$\lim(b_n) = \inf\{b_n : n \in \mathbb{N}\}$$

Recall Theorem 3.2.5:

Theorem. If $X = (x_n)$ and $Y = (y_n)$ are convergent sequences of real numbers and if $x_n \leq y_n \; \forall \; n \in \mathbb{N}$, then $\lim(x_n) \leq \lim(y_n)$.

Also, recall the *Nested Intervals Property*:

Theorem. If $I_n = [a_n, b_n], \; n \in \mathbb{N}$, is a nested sequence of closed bounded intervals, then there exists a number $\xi \in \mathbb{R}$ s.t. $\xi \in I_n \; \forall \; n \in \mathbb{N}$.

Note that we have a nested sequence of closed, bounded intervals: $[a_n, b_n], \; n \in \mathbb{N}$. Since we showed that $\lim(a_n) \leq \lim(b_n)$, (and we are given that (a_n) is increasing and (b_n) is decreasing), we know that there exists ξ such that

$$\lim(a_n) \leq \xi \leq \lim(b_n)$$

which means that $\xi \in [a_n, b_n], \; \forall \; n \in \mathbb{N}$.

(b) $a_1 = 1, \; a_{n+1} = \frac{a_n^2+5}{2a_n}$

The first 5 terms of this sequence are $1, 3, \frac{7}{3}, \frac{47}{21}, \frac{2207}{987}, \cdots \approx 1, 3, 2.3333, 2.2381, 2.2361, \ldots$. This is a decreasing sequence.

First, we must find the possible limits (fixed points) of the sequence. So,

$$\begin{aligned} a &= \frac{a^2+5}{2a} \\ 2a^2 &= a^2 + 5 \\ a^2 &= 5 \\ a &= \pm\sqrt{5} \end{aligned}$$

Since we're given that $a_1 = 1$, we know that the most likely lower bound will be $\sqrt{5}$.

Now we want to show that (a_n) is bounded below by $\sqrt{5}$.

Proof. We want to show that $a_n \geq \sqrt{5}, \; \forall \; n \in \mathbb{N}$. We prove it by method of mathematical induction.

Basis Step: Since $1 \geq \sqrt{5}$, we have that $a_1 \geq \sqrt{5}$

Inductive Step: Assume that $a_n \geq \sqrt{5} \; \forall \; n \in \mathbb{N}$.

Show: We want to show that $a_{n+1} \geq \sqrt{5} \; \forall \; n \in \mathbb{N}$. So,

$$a_{n+1} = \frac{a_n^2+5}{2a_n}$$

$$\begin{aligned}
(a_n - \sqrt{5})^2 &\geq 0 \\
a_n^2 - 2\sqrt{5}a_n + 5 &\geq 0 \\
a_n^2 + 5 &\geq 2\sqrt{5}a_n \\
&\Downarrow \\
\frac{a_n^2+5}{2a_n} &\geq \frac{2\sqrt{5}a_n}{2a_n} \\
\frac{a_n^2+5}{2a_n} &\geq \sqrt{5} \\
a_{n+1} &\geq \sqrt{5}
\end{aligned}$$

Thus we have that (a_n) is bounded below by $\sqrt{5}$. ■

Now we must show that (a_n) is monotone decreasing.

Proof. We want to show that (a_n) is monotone decreasing; that is, we want to show that $(a_2 \geq a_3 \geq \cdots \geq a_n)$, $\forall\ n \geq 2$. We prove it by method of mathematical induction.

Basis Step: Since $3 \geq \frac{7}{3}$, we have that $a_2 \geq a_3$.

Inductive Step: Assume that $a_n \geq a_{n+1}$, $\forall\ n \geq 2$.

Show: We want to show that $a_{n+2} \leq a_{n+1}$, $\forall\ n \geq 2$.
So,

$$a_{n+2} = \frac{a_{n+1}^2+5}{2a_{n+1}} \leq \frac{a_n^2+5}{2a_n}$$

Since we have:

$$\begin{aligned}
a_{n+1} &\geq \sqrt{5}, && \text{by the previous proof of boundedness} \\
a_{n+1}^2 &\geq 5
\end{aligned}$$

We can equivalently write the inequality as

$$\frac{a_{n+1}^2+5}{2a_{n+1}} \leq \frac{a_{n+1}^2+a_{n+1}^2}{2a_{n+1}} = a_{n+1}$$

Thus we have that (a_n) is monotone decreasing. ■

Since (a_n) is both monotone decreasing and bounded, we have

$$\begin{aligned}
\lim(a_n) &= \inf\{a_n : n \in \mathbb{N}\} \\
&= \sqrt{5}
\end{aligned}$$

(c) $a_1 = 5,\ a_{n+1} = \sqrt{4 + a_n}$

The first 5 terms of this sequence are 5, 3, $\sqrt{7}$, $\frac{\sqrt{14}}{2} + \frac{\sqrt{2}}{2}$, $\frac{\sqrt{2 \cdot (\sqrt{14} + \sqrt{2} + 8)}}{2}$, $\ldots$, $\approx$ 5, 3, 2.64575131106, 2.57793547457, 2.5647486182, ... This sequence is decreasing.

First, we must find the possible limits (fixed points) of the sequence. So,

$$\begin{aligned}
a &= \sqrt{4+a} \\
\sqrt{4+a} &= a \\
4 + a &= a^2 \\
-a^2 + a + 4 &= 0 \\
a^2 - a - 4 &= 0 \\
a^2 - a &= 4 \\
a^2 - a + \frac{1}{4} &= 4 + \frac{1}{4} \\
a^2 - a + \frac{1}{4} &= \frac{17}{4} \\
(a - \frac{1}{2})^2 &= \frac{17}{4} \\
a - \frac{1}{2} &= \pm\frac{\sqrt{17}}{2}
\end{aligned}$$

So we have that $a = \frac{1}{2} + \frac{\sqrt{17}}{2}$, or $a = \frac{1}{2} - \frac{\sqrt{17}}{2}$. We must now check these solutions for correctness; so,

$$\begin{aligned}
a \Rightarrow \frac{1}{2} - \frac{\sqrt{17}}{2} &= \frac{1}{2}\left(1 - \sqrt{17}\right) \\
&\approx -1.56155
\end{aligned}$$

$$\begin{aligned}
\sqrt{a+4} &= \sqrt{\left(\frac{1}{2} - \frac{\sqrt{17}}{2}\right) + 4} \\
&= \frac{\sqrt{9 - \sqrt{17}}}{\sqrt{2}} \\
&\approx 1.56155
\end{aligned}$$

Thus, this solution is incorrect. Now we must validate that $a = \frac{1}{2} + \frac{\sqrt{17}}{2}$ is correct.

So,

$$a \Rightarrow \frac{1}{2} + \frac{\sqrt{17}}{2} = \frac{1}{2}\left(1 + \sqrt{17}\right)$$
$$\approx 2.56155$$

$$\begin{aligned}\sqrt{a+4} &= \sqrt{\left(\frac{\sqrt{17}}{2} + \frac{1}{2}\right) + 4} \\ &= \frac{\sqrt{9+\sqrt{17}}}{\sqrt{2}} \\ &\approx 2.56155\end{aligned}$$

Thus $a = \frac{1}{2} + \frac{\sqrt{17}}{2}$ is a correct solution.

Now we want to show that (a_n) is bounded below by $\frac{1}{2} + \sqrt{17}$.

Proof. We want to show that $a_n \geq \frac{1}{2} + \frac{\sqrt{17}}{2}$, $\forall\ n \in \mathbb{N}$, by the definition of a lower bound. We prove this by method of mathematical induction.

Basis Step: Since $5 \geq \frac{1}{2} + \frac{\sqrt{17}}{2}$, we have that $a_1 \geq \frac{1}{2} + \frac{\sqrt{17}}{2}$.

Inductive Step: Assume $a_n \geq \frac{1}{2} + \frac{\sqrt{17}}{2}$, $\forall\ n \in \mathbb{N}$.

Show: We now want to show that $a_{n+1} \geq \frac{1}{2} + \frac{\sqrt{17}}{2}\ \forall\ n \in \mathbb{N}$. So,

$$\begin{aligned}
a_{n+1} &= \sqrt{4 + a_n}, && \text{by the definition of the sequence} \\
&\geq \sqrt{4 + \left(\frac{1}{2} + \frac{\sqrt{17}}{2}\right)}, && \text{by the inductive hypothesis} \\
&\geq \sqrt{\frac{8}{2} + \frac{1}{2} + \frac{\sqrt{17}}{2}} \\
&\geq \sqrt{\frac{9 + \sqrt{17}}{2}} \\
&\geq \sqrt{\frac{1}{2}\left(9 + \sqrt{17}\right)} \\
&\geq \sqrt{\frac{1}{4} + \frac{\sqrt{17}}{2} + \frac{17}{4}}, && \text{by expressing } \frac{9+\sqrt{17}}{2} \text{ as a square} \\
&\geq \sqrt{\frac{1 + 2\sqrt{17} + 17}{4}} \\
&\geq \sqrt{\frac{1 + 2\sqrt{17} + (\sqrt{17})^2}{4}} \\
&\geq \sqrt{\frac{(\sqrt{17} + 1)^2}{4}} \\
&\geq \sqrt{\frac{1}{4}\left(1 + \sqrt{17}\right)^2} \\
&\geq \frac{\sqrt{(1 + \sqrt{17})^2}}{\sqrt{4}} \\
&\geq \frac{\sqrt{17} + 1}{2} \\
&\geq \frac{1}{2} + \frac{\sqrt{17}}{2}
\end{aligned}$$

Thus we have that $a_{n+1} \geq \frac{1}{2} + \frac{\sqrt{17}}{2}\ \forall\ n \in \mathbb{N}$. ■

Now, we want to show that (a_n) is monotone decreasing; that is, we want to show that $(a_1 \geq a_2 \geq \cdots \geq a_n)$.

Proof. We want to show that $(a_1 \geq a_2 \geq \cdots \geq a_n)$, $\forall\ n \in \mathbb{N}$. We prove this by method of mathematical induction.

Basis Step: Since $5 \geq 3$, we have that $a_1 \geq a_2$.

Inductive Step: Assume $a_n \geq a_{n+1} \ \forall \ n \in \mathbb{N}$.

Show: We want to show that $a_{n+1} \geq a_{n+2} \ \forall \ n \in \mathbb{N}$. So,

$$\begin{aligned} a_{n+2} &= \sqrt{4 + a_{n+1}} && \text{by the definition of the sequence} \\ &\leq \sqrt{4 + a_n} && \text{by the inductive hypothesis} \\ &= a_{n+1} \end{aligned}$$

Thus we have that $a_{n+1} \geq a_{n+2} \ \forall \ n \in \mathbb{N}$. ■

Since (a_n) is both bounded and monotone decreasing, by the *Monotone Convergence Theorem*, we have that (a_n) converges. Also by the *Monotone Sequence Property*, we have that (a_n) converges to the following:

$$\begin{aligned} \lim(a_n) &= \inf\{a_n : n \in \mathbb{N}\} \\ &= \frac{1}{2} + \frac{\sqrt{17}}{2} \approx 2.56155281281 \end{aligned}$$

2. **(a)** Show $a_n = \frac{3 \cdot 5 \cdot 7 \cdot \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots (2n)}$ converges to A where $0 \leq A < 1/2$.

TODO

(b) Show $b_n = \frac{2 \cdot 4 \cdot 6 \ldots (2n)}{3 \cdot 5 \cdot 7 \ldots (2n+1)}$ converges to B where $0 \leq B < 2/3$.

TODO

3. **Section 3.4**

1) Give an example of an unbounded sequence that has a convergent subsequence.

Let $a_n := (-1)^n$. This sequence diverges since it oscillates between 1 and -1, however if we define $b_n := (a_{2n})$; that is, let (b_n) be the sequence of all terms of (a_n) such that n is even. Thus, (b_n) is a subsequence of (a_n), but (b_n) converges since $a_2 = 1, a_4 = 1, \ldots, a_{2n} = 1$. Thus while (a_n) is a divergent sequence, the subsequence (b_n) of (a_n) converges for all $n \in \mathbb{N}$.

3) Let (f_n) be the Fibonacci sequence of Example 3.1.2(d), and let $x_n := f_{n+1}/f_n$. Given that $\lim(x_n) = L$ exists, determine the value of L.

We can rewrite (x_n) as follows:

$$\begin{aligned} x_n &= \frac{f_{n+1}}{f_n} \\ &= \frac{f_n + f_{n-1}}{f_n} \\ &= 1 + \frac{f_{n-1}}{f_n} \\ &= 1 + \frac{\frac{1}{f_n}}{f_{n-1}} \\ &= 1 + \frac{1}{x_{n-1}} \end{aligned}$$

Since we're given that $L = \lim(x_n)$ exists and since we just showed that it's equal to $\lim(x_{n-1})$, we get the following:

$$\begin{aligned} x_n &= 1 + \frac{1}{x_{n-1}} \qquad \Big| \lim \\ \lim(x_n) &= 1 + \frac{1}{\lim(x_{n-1})} \\ L &= 1 + \frac{1}{L} \qquad \Big| \cdot L \\ L^2 &= L + 1 \\ L^2 - L - 1 &= 0 \\ L_{1,2} &= \frac{1 \pm \sqrt{1 - 4 \cdot 1 \cdot (-1)}}{2} \\ L_1 &= \frac{1 - \sqrt{5}}{2} < 0 \\ L_2 &= \frac{1 + \sqrt{5}}{2} > 0 \end{aligned}$$

Now, since $f_n > 0 \Rightarrow x_n > 0 \Rightarrow L > 0$, we can infer that the proper limit is

$$L = \frac{1 + \sqrt{5}}{2}$$

4a) Show that the sequence $(1 - (-1)^n + 1/n)$ converges.

Let $(x_n) := (1 - (-1)^n + 1/n)$. Let $(z_n) = (x_{2n})$, and $(w_n) = (x_{2n-1})$ be subsequence of (x_n). Then (z_n) is the subsequence of all terms of (x_n) such that n is even, and (w_n) is the subsequence of all terms of (x_n) such that n is odd.

These subsequences yield the following:

$$z_n = x_{2n} = 1 - (-1)^{2n} + \frac{1}{2n} = 1 - 1 + \frac{1}{2n} = \frac{1}{2n}$$

$$w_n = x_{2n-1} = 1 - (-1)^{2n-1} + \frac{1}{2n-1} = 1 + 1 + \frac{1}{2n-1} = 2 + \frac{1}{2n-1}$$

Now, if we take the limit of each sequence as $n \to \infty$ yields

$$\lim_{n\to\infty} (z_n) = 0 \neq 2 = \lim_{n\to\infty} (w_n)$$

Recall Theorem 3.4.5 *Divergence Criteria*:

Theorem. If a sequence $X = (x_n)$ of real numbers has either of the following properties, then X is divergent.

i. X has two convergent subsequences $X' = (x_{n_k})$ and $X'' = (x_{r_k})$ whose limits are not equal.

ii. X is unbounded

Thus by the *Divergence Criteria*, we have that since (z_n) and (w_n) satisfy the first property of the *Divergence Criteria*, we can conclude that the sequence (x_n) is divergent.

16) Give an example to show that Theorem 3.4.9 fails if the hypothesis that X is a bounded sequences is dropped.

Recall *Theorem 3.4.9*:

Theorem. If (x_n) is a bounded sequence of real numbers, then the following statements for a real number x^* are equivalent:

i. $x^* = \limsup(x_n)$.

ii. If $\varepsilon > 0$, there are at most a finite number of $n \in \mathbb{N}$ such that $x^* + \varepsilon < x_n$, but an infinite number of $n \in \mathbb{N}$ such that $x^* - \varepsilon < x_n$.

iii. If $u_m = \sup\{x_n : n \geq m\}$, then $x^* = \inf\{u_m : m \in \mathbb{N}\} = \lim(u_m)$.

iv. If S is the set of subsequential limits of (x_n), then $x^* = \sup(S)$.

Consider the sequence $((-1)^n n)$. We note that any subsequence of this sequence is unbounded and thus this sequence has no convergent subsequence. Due to this, all of the conditions of *Theorem 3.4.9* are satisfied vacuously, save the condition concerning boundedness. However, this sequence doesn't converge, but both oscillates and diverges towards ∞ and $-\infty$. Thus if the boundedness criterion of the theorem is dropped, this theorem fails.

18) Show that if (x_n) is a bounded sequence, then (x_n) converges if and only if $\limsup(x_n) = \liminf(x_n)$.

Proof. Let (x_n) be a bounded sequence. We want to show that (x_n) converges if and only if $\limsup(x_n) = \liminf(x_n)$.

TODO $\blacksquare$

19) Show that if (x_n) and (y_n) are bounded sequences, then

$$\limsup(x_n + y_n) \leq \limsup(x_n) + \limsup(y_n).$$

Give an example in which the two sides are not equal.

TODO

4. **(a)** Show that $x_n = e^{\sin(5n)}$ has a convergent subsequence.

(b) Give an example of a bounded sequence with three subsequences converging to three different numbers.

(c) Give an example of a sequence x_n with $\limsup x_n = 5$ and $\limsup x_n = -3$.

(d) Let $\limsup x_n = 2$. True or False: if n is sufficiently large, then $x_n > 1.99$.

(e) Compute the infimum, supremum, limit infimum, and limit supremum for $a_n = 3 - (-1)^n - (-1)^n/n$.

5. Prove or justify, if true. Provide a counterexample, if false.

(a) If a_n and b_n are strictly increasing, then $a_n + b_n$ is strictly increasing.

(b) If a_n and b_n are strictly increasing, then $a_n \cdot b_n$ is strictly increasing.

(c) If a_n and b_n are monotonic, then $a_n + b_n$ is monotonic.

(d) If a_n and b_n are monotonic, then $a_n \cdot b_n$ is monotonic.

(e) If a monotone sequence is bounded, then it is convergent.

(f) If a bounded sequence is monotone, then it is convergent.

(g) If a convergent sequence is monotone, then it is bounded.

(h) If a convergent sequence is bounded, then it is monotone.