Real Analysis Homework 5

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1. For the following sequences, i) write out the first 5 terms, ii) Use the Monotone Sequence Property to show that the sequences converges.

(a) Section 3.3

2) Let $x_1 > 1$ and $x_{n+1} := 2 - 1/x_n$ for $n \in \mathbb{N}$. Show that (x_n) is bounded and monotone. Find the limit.

The first five terms of this sequence are $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \ldots$.
This sequence appears to be decreasing. Thus, we attempt to find the possible limits of this sequence.

$$\begin{aligned} x &= 2 - \frac{1}{x} \\ x - 2 &= -\frac{1}{x} \\ x^2 - 2x &= -1 \\ x^2 - 2x + 1 &= 0 \\ (x-1)^2 &= 0 \end{aligned}$$

So we have that $x = 1$

Proof. Let $x_1 > 1$ and $x_{n+1} := 2 - 1/x_n$ for $n \in \mathbb{N}$. We want to show that (x_n) is monotone decreasing; that is, we want to show that $x_{n+1} \leq x_n$, by the definition of monotone decreasing. We prove it by method of mathematical induction.

Basis Step: Let $n = 1$. Then we have

$$\begin{aligned} x_{n+1} &\leq x_n \\ x_{1+1} &\leq x_1 \\ x_2 &\leq x_1 \\ \frac{3}{2} &\leq 2, \end{aligned}$$

which is true. So $a_2 \leq a_1$.

Inductive Step: Assume that $x_{n+1} \leq x_n, \ \forall \ n \in \mathbb{N}$.

We now want to show that $x_{n+2} \leq x_{n+1}$.
So,

$$\begin{aligned} x_{n+2} = 2 - \frac{1}{x_{n+1}} &\leq 2 - \frac{1}{x_n}, && \text{since } x_{n+1} \leq x_n \\ &= x_{n+1}, && \text{by the definition of the sequence.} \end{aligned}$$

Thus we have that $x_{n+2} \leq x_{n+1}$. ■

Proof. Now, we must show that x_n is bounded below; that is, we want to show that $x_n \geq 1, \ \forall \ n \in \mathbb{N}$. We prove it by method of mathematical induction.

Basis Step: Let $n = 1$. Then we have

$$x_1 = 2 \geq 1$$

which is true.

Inductive Step: Assume $x_n \geq 1$.

We now want to show that $x_{n+1} \geq 1$.
So, ■

3) Let $x_1 > 1$ and $x_{n+1} := 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$. Show that (x_n) is decreasing and bounded below by 2. Find the limit.

7) Let $x_1 := a > 0$ and $x_{n+1} := x_n + 1/x_n$ for $n \in \mathbb{N}$. Determine whether (x_n) converges or diverges.

8) Let (a_n) be an increasing sequence, (b_n) be a decreasing sequence, and assume that $a_n \leq b_n$ for all $n \in \mathbb{N}$. Show that $\lim(a_n) \leq \lim(b_n)$, and thereby deduce the Nested Intervals Property 2.5.2 from the Monotone Convergence Theorem 3.3.2.

(b) $a_1 = 1, \ a_{n+1} = \frac{a_n^2+5}{2a_n}$

(c) $a_1 = 5, \ a_{n+1} = \sqrt{4 + a_n}$

2. (a) Show $a_n = \frac{3 \cdot 5 \cdot 7 \cdot \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots (2n)}$ converges to A where $0 \leq A < 1/2$.

(b) Show $b_n = \frac{2 \cdot 4 \cdot 6 \ldots (2n)}{3 \cdot 5 \cdot 7 \ldots (2n+1)}$ converges to B where $0 \leq B < 2/3$.

3. Section 3.4

1) Give an example of an unbounded sequence that has a convergent subsequence.

3) Let (f_n) be the Fibonacci sequence of Example 3.1.2(d), and let $x_n := f_{n+1}/f_n$. Given that $\lim(x_n) = L$ exists, determine the value of L.

4a) Show that the sequence $(1 - (-1)^n + 1/n)$ converges.

16) Give an example to show that Theorem 3.4.9 fails if the hypothesis that X is a bounded sequences is dropped.

18) Show that if (x_n) is a bounded sequence, then (x_n) converges if and only if $\limsup(x_n) = \liminf(x_n)$.

19) Show that if (x_n) and (y_n) are bounded sequences, then

$$\limsup(x_n + y_n) \leq \limsup(x_n) + \limsup(y_n).$$

Give an example in which the two sides are not equal.

4. **(a)** Show that $x_n = e^{\sin(5n)}$ has a convergent subsequence.

(b) Give an example of a bounded sequence with three subsequences converging to three different numbers.

(c) Give an example of a sequence x_n with $\limsup x_n = 5$ and $\limsup x_n = -3$.

(d) Let $\limsup x_n = 2$. True or False: if n is sufficiently large, then $x_n > 1.99$.

(e) Compute the infimum, supremum, limit infimum, and limit supremum for $a_n = 3 - (-1)^n - (-1)^n/n$.

5. **(a)** If a_n and b_n are strictly increasing, then $a_n + b_n$ is strictly increasing.

(b) If a_n and b_n are strictly increasing, then $a_n \cdot b_n$ is strictly increasing.

(c) If a_n and b_n are monotonic, then $a_n + b_n$ is monotonic.

(d) If a_n and b_n are monotonic, then $a_n \cdot b_n$ is monotonic.

(e) If a monotone sequence is bounded, then it is convergent.

(f) If a bounded sequence is monotone, then it is convergent.

(g) If a convergent sequence is monotone, then it is bounded.

(h) If a convergent sequence is bounded, then it is monotone.