Real Analysis II Homework 8

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1. Section 9.2

2. (c) Establish the convergence or divergence of the series whose nth term is $n!/n^n$.

Since

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} \\
&= \frac{(n+1)! \cdot n^n}{(n+1)^{n+1} \cdot n!} \\
&= \frac{n^n}{(n+1)^n} \\
&\Downarrow \\
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \frac{1}{\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^n} \\
&= \frac{1}{e} \\
&< 1
\end{aligned}$$

By the *Ratio Test*, and by *Corollary 9.2.5*, we have that $\sum \frac{n!}{n^n}$ converges.

5. Show that the series $1/1^2 + 1/2^3 + 1/3^2 + 1/4^3 + \dots$ is convergent, but that both the Ratio and the Root Tests fail to apply.

We notice that the specified series yields $S = \sum a_n$, where

$$a_{2n} = \frac{1}{(2n)^3}, \quad \text{and} \quad a_{2n-1} = \frac{1}{(2n-1)^2}, \quad \text{for} \quad n \in \mathbb{N}$$

First we show that the *Ratio Test* fails. To do so, we must consider two cases:

$$\begin{aligned}
\left|\frac{a_{2n+1}}{a_{2n}}\right| &= \frac{\frac{1}{(2n+1)^2}}{\frac{1}{(2n)^3}} \\
&= \frac{8n^3}{4n^2+4n+1} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}} \\
&= \frac{8}{\frac{4}{n}+\frac{4}{n^2}+\frac{1}{n^3}} \\
&\Downarrow
\end{aligned}$$

$$\lim_{n\to\infty} \frac{8}{\frac{4}{n}+\frac{4}{n^2}+\frac{1}{n^3}} = \infty$$

and

$$\begin{aligned}
\left|\frac{a_{2n}}{a_{2n-1}}\right| &= \frac{\frac{1}{(2n)^3}}{\frac{1}{(2n-1)^2}} \\
&= \frac{4n^2-4n+1}{8n^3} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}} \\
&= \frac{\frac{4}{n}-\frac{4}{n^2}+\frac{1}{n^3}}{8} \\
&\Downarrow
\end{aligned}$$

$$\lim_{n\to\infty} \frac{\frac{4}{n}-\frac{4}{n^2}+\frac{1}{n^3}}{8} = 0$$

And thus we can see that the *Ratio Test* is ineffective on this series.

As for the *Root Test*, we must also consider two cases:

$$|a_{2n}|^{\frac{1}{2n}} = \left(\frac{1}{8n^3}\right)^{\frac{1}{2n}} = \frac{1}{8^{\frac{2}{n}}} \cdot \left(\frac{1}{n^{\frac{2}{n}}}\right)^3 \implies \lim_{n\to\infty} \frac{1}{8^{\frac{2}{n}}} \cdot \left(\frac{1}{n^{\frac{2}{n}}}\right)^3 = \frac{1}{1} \cdot \left(\frac{1}{1}\right)^3 = 1$$

and

$$|a_{2n-1}|^{\frac{1}{2n-1}} = \left(\frac{1}{(2n-1)^2}\right)^{\frac{1}{2n-1}} = \left(\frac{1}{2n-1}\right)^{\frac{2}{2n-1}} \implies \lim_{n\to\infty} \left(\frac{1}{2n-1}\right)^{\frac{2}{2n-1}} = 1$$

And thus we see that the *Root Test* is ineffective on this series as well.

Now, we'll show that the series does converge by the *Comparison Test*:

We notice that

$$\begin{aligned}
a_{2n} &= \frac{1}{8n^3} < \frac{1}{n^3} < \frac{1}{n^2} \\
a_{2n-1} &= \frac{1}{(2n-1)^2} < \frac{1}{n^2}
\end{aligned}$$

and thus by the *Comparison Test*, we have that since $\frac{1}{n^2}$ is a convergent p-series with $p = 2 > 1$, the series $\sum a_n$ must also converge.

7. Discuss the series whose nth term is

(a) $\dfrac{n!}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+1)}$

By the *Ratio Test*, we have:

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \frac{\dfrac{(n+1)!}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2(n+1)+1)}}{\dfrac{n!}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+1)}} \\
&= \frac{(n+1)! \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+1)}{n! \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+1) \cdot (2n+3)} \\
&= \frac{n+1}{2n+3} \leq \frac{n+1}{2n+2} \\
&= \frac{n+1}{2(n+1)} \\
&= \frac{1}{2} \\
&< 1
\end{aligned}$$

And thus by the *Ratio Test*, we have that the series is absolutely convergent.

(b) $\dfrac{(n!)^2}{(2n)!}$

By the *Ratio test*, we have:

$$
\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \frac{\dfrac{\big((n+1)!\big)^2}{\big(2(n+1)\big)!}}{\dfrac{(n!)^2}{(2n)!}} \\
&= \frac{(n+1)!\cdot(n+1)!\cdot(2n)!}{(2n+2)!\cdot n!\cdot n!} \\
&= \frac{(n+1)(n+1)}{(2n+1)(2n+2)} \\
&= \frac{n+1}{2(2n+1)} \\
&= \frac{n+1}{4n+2} \\
&\leq \frac{n+1}{4n} \\
&= \frac{1}{4}+\frac{1}{4n} \\
&\leq \frac{1}{4}+\frac{1}{4} \\
&= \frac{1}{2} \\
&< 1
\end{aligned}
$$

And thus by the *Ratio Test*, we have that the series is absolutely convergent.

(c) $\dfrac{2\cdot 4\cdot\cdots\cdot(2n)}{3\cdot 5\cdot\cdots\cdot(2n+1)}$

By the *Ratio Test*, we have:

$$
\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \frac{\dfrac{2\cdot 4\cdot\cdots\cdot(2n)\cdot(2n+2)}{3\cdot 5\cdot\cdots\cdot(2n+1)\cdot(2n+3)}}{\dfrac{2\cdot 4\cdot\cdots\cdot(2n)}{3\cdot 5\cdot\cdots\cdot(2n+1)}} \\
&= \frac{2n+2}{2n+3} \\
&\Downarrow
\end{aligned}
$$

$$
\lim_{n\to\infty}\left|\frac{2n+2}{2n+3}\right| = 1
$$

Thus by *Corollary 9.2.5*, the *Ratio Test* is ineffective on this series.

By *Raabe's Test*, we have:

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \frac{2n+2}{2n+3} \\
&= \frac{(2n+3)-1}{2n+3} \\
&= 1 - \frac{1}{2n+3} \\
&\geq 1 - \frac{1}{2n} \\
&= 1 - \frac{\frac{1}{2}}{n}
\end{aligned}$$

Thus by *Raabe's Test*, since $a = \frac{1}{2}$, we have that the series is divergent.

(d) $\dfrac{2 \cdot 4 \cdot \cdots \cdot (2n)}{5 \cdot 7 \cdot \cdots \cdot (2n+3)}$

By the *Ratio Test*, we have:

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \frac{\frac{2 \cdot 4 \cdot \cdots \cdot (2n) \cdot (2n+2)}{5 \cdot 7 \cdot \cdots \cdot (2n+3) \cdot (2n+5)}}{\frac{2 \cdot 4 \cdot \cdots \cdot (2n)}{5 \cdot 7 \cdot \cdots \cdot (2n+3)}} \\
&= \frac{2n+2}{2n+5} \\
&\Downarrow \\
\lim_{n\to\infty} \frac{2n+2}{2n+5} &= 1
\end{aligned}$$

Thus by *Corollary 9.2.5*, the *Ratio Test* is ineffective on this series.

By *Corollary 9.2.9*, we have:

$$\begin{aligned}
a &= \lim_{n\to\infty} \left(n \left(1 - \left|\frac{a_{n+1}}{a_n}\right| \right) \right) \\
&= \lim_{n\to\infty} \left(n \left(1 - \frac{2n+2}{2n+5} \right) \right) \\
&= \lim_{n\to\infty} \left(n \cdot \frac{3}{2n+5} \right) \\
&= \lim_{n\to\infty} \left(\frac{3}{2 + \frac{5}{n}} \right) \\
&= \frac{3}{2}
\end{aligned}$$

Thus by *Corollary 9.2.9*, since $a = \frac{3}{2} > 1$, we have that the series is absolutely convergent.

2. Section 9.3

1. Test the following series for convergence and for absolute convergence:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+1}$

By the *Alternating Series Test*, we have that $\displaystyle\lim_{n\to\infty} \frac{1}{n^2+1} = 0$. And by the *Limit Comparison Test*, we notice that $\frac{1}{n^2+1}$ looks like $\frac{1}{n^2}$, which we note is a convergent p-series with $p = 2 > 1$, which yields:

$$\lim_{n\to\infty} \frac{\frac{1}{n^2+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \neq 0$$

And thus since $\sum \frac{1}{n^2}$ is convergent, by the *Limit Comparison Test*, we have that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+1}$ is absolutely convergent.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$

By the *Alternating Series Test*, we have that $\displaystyle\lim_{n\to\infty} \frac{1}{n+1} = 0$, and thus the series is convergent. And by the *Limit Comparison Test*, we note that the series looks like $\sum \frac{1}{n}$, which we note is a harmonic series and thus diverges, which yields

$$\lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$$

And since $\sum \frac{1}{n}$ is a harmonic series and thus diverges, since the limit is not equal to 0, we have that by the *Limit Comparison Test*, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$ is conditionally convergent.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n+2}$

By the *Alternating Series Test*, we have that $\displaystyle\lim_{n\to\infty} \frac{n}{n+2} = 1 \neq 0$, and thus the series is divergent.

(d) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{n}$

By the *Alternating Series Test*, we have that $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n} = 0$, which yields that the series is convergent. And by the *Integral Test*, we have that

$$\int_1^\infty \frac{\ln n}{n}\,dn = \int_1^\infty \frac{u}{n}\cdot n\,du = \int_1^\infty u\,du = \frac{u^2}{2}\bigg|_1^\infty = \frac{(\ln n)^2}{2}\bigg|_1^\infty = \frac{(\ln\infty)^2}{2} - \frac{(\ln 1)^2}{2}$$

$$= \infty - 0 = \infty$$

Thus by the *Integral Test*, we have that this series is divergent. Thus the series is conditionally convergent.

3. Give an example to show that the Alternating Series Test 9.3.2 may fail if (z_n) is not a decreasing sequence.

Let $\sum a_n$ be the series defined as

$$\left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} + \cdots + \frac{1}{n} - \frac{1}{2n} + \dots\right)$$

Let A_n be the partial sums of the series $\sum a_n$. Then since a_n is an alternating sequence that converges to 0 and isn't decreasing, we have

$$A_{2n} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{2n}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{2n}\right)$$

As A_{2n} diverges, we have that A_n diverges and thus the series $\sum a_n$ is divergent.

5. Consider the series

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + + - - \dots,$$

where the signs come in pairs. Does it converge?

We notice that $\frac{1}{n}$ is a monotone decreasing sequence that converges to 0. Then we notice that the series $\sum\limits_{n=1}^{\infty} a_n$ where for every $n \in \mathbb{N}$, we have $a_1 = 1, a_{4n} = 1, a_{4n-1} = -1 = a_{4n+1}$. Now, let $s_n = a_1 + a_2 + \cdots + a_n$. Then $s_{2n} = 0$ and $s_{2n+1} = \pm 1$. This yields that $|s_n| \le 1$. By *Dirichlet's Test*, we have that $\sum\limits_{n=1}^{\infty} a_n$ is convergent. Thus we have

$$\sum_{n=1}^{\infty}\frac{a_n}{n} = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + + - - \dots$$

is also convergent.

3. Give an example of a series $\sum a_n$ that consists of nonzero terms with $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$ for each of the following conditions:

(a) $\sum a_n$ converges absolutely

Consider Problem 7d from Section 9.2: $\displaystyle\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot \cdots \cdot (2n)}{5 \cdot 7 \cdot \cdots \cdot (2n+3)}$

By the *Ratio Test*, we have:

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \frac{\dfrac{2 \cdot 4 \cdot \cdots \cdot (2n) \cdot (2n+2)}{5 \cdot 7 \cdot \cdots \cdot (2n+3) \cdot (2n+5)}}{\dfrac{2 \cdot 4 \cdot \cdots \cdot (2n)}{5 \cdot 7 \cdot \cdots \cdot (2n+3)}} \\
&= \frac{2n+2}{2n+5} \\
&\Downarrow \\
\lim_{n\to\infty} \frac{2n+2}{2n+5} &= 1
\end{aligned}$$

Thus by *Corollary 9.2.5*, the *Ratio Test* is ineffective on this series.

By *Corollary 9.2.9*, we have:

$$\begin{aligned}
a &= \lim_{n\to\infty} \left(n \left(1 - \left| \frac{a_{n+1}}{a_n} \right| \right) \right) \\
&= \lim_{n\to\infty} \left(n \left(1 - \frac{2n+2}{2n+5} \right) \right) \\
&= \lim_{n\to\infty} \left(n \cdot \frac{3}{2n+5} \right) \\
&= \lim_{n\to\infty} \left(\frac{3}{2 + \frac{5}{n}} \right) \\
&= \frac{3}{2}
\end{aligned}$$

Thus by *Corollary 9.2.9*, since $a = \frac{3}{2} > 1$, we have that the series is absolutely convergent.

(b) $\sum a_n$ converges conditionally

Consider problem 1b from Section 9.3: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$

By the *Ratio Test*, we have:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{\dfrac{1}{n+3}}{\dfrac{1}{n+1}} \\
&= \lim_{n\to\infty}\frac{n+1}{n+3} \\
&= 1
\end{aligned}$$

Thus by *Corollary 9.2.5*, the *Ratio Test* is ineffective on this series.

By the *Alternating Series Test*, we have that $\lim\limits_{n\to\infty}\dfrac{1}{n+1}=0$, and thus the series is convergent. And by the *Limit Comparison Test*, we note that the series looks like $\sum\frac{1}{n}$, which we note is a harmonic series and thus diverges, which yields

$$\lim_{n\to\infty}\frac{\dfrac{1}{n+1}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{n}{n+1}=1\neq 0$$

And since $\sum\frac{1}{n}$ is a harmonic series and thus diverges, since the limit is not equal to 0, we have that by the *Limit Comparison Test*, the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+1}$ is conditionally convergent.

(c) $\sum a_n$ diverges.

Consider Problem 7c from Section 9.2: $\displaystyle\sum_{n=1}^{\infty}\frac{2\cdot 4\cdot\cdots\cdot(2n)}{3\cdot 5\cdot\cdots\cdot(2n+1)}$

By the *Ratio Test*, we have:

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \frac{\dfrac{2\cdot 4\cdot\cdots\cdot(2n)\cdot(2n+2)}{3\cdot 5\cdot\cdots\cdot(2n+1)\cdot(2n+3)}}{\dfrac{2\cdot 4\cdot\cdots\cdot(2n)}{3\cdot 5\cdot\cdots\cdot(2n+1)}} \\
&= \frac{2n+2}{2n+3} \\
&\Downarrow \\
\lim_{n\to\infty}\left|\frac{2n+2}{2n+3}\right| &= 1
\end{aligned}$$

Thus by *Corollary 9.2.5*, the *Ratio Test* is ineffective on this series.

By *Raabe's Test*, we have:

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{2n+2}{2n+3}\\ &= \frac{(2n+3)-1}{2n+3}\\ &= 1-\frac{1}{2n+3}\\ &\geq 1-\frac{1}{2n}\\ &= 1-\frac{\frac{1}{2}}{n}\end{aligned}$$

Thus by *Raabe's Test*, since $a=\frac{1}{2}$, we have that the series is divergent.

4. Prove or justify, if true. Provide a counterexample, if false.

(a) If $\sum |a_n|$ diverges, then $\sum a_n$ is conditionally convergent.

This is a false statement. Consider the sequence $a_n = (0, \frac{1}{2}, 0, \frac{1}{4}, 0, \dots)$. we note that all of the terms are greater than or equal to 0. We also note that the sequence can be defined piecewise as follows:

$$a_n := \begin{cases} 0, & n \text{ is odd} \\ \frac{1}{n}, & n \text{ is even} \end{cases}$$

for all $n \in \mathbb{N}$. By this definition, we have

$$\begin{aligned}\sum_{n=1}^{\infty} |a_n| &= \sum_{n=1}^{\infty} a_n\\ &= 0+\frac{1}{2}+0+\frac{1}{4}+0+\frac{1}{6}+\dots\\ &= \frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\dots\\ &= \sum_{n=1}^{\infty}\frac{1}{2n}\\ &= \infty\end{aligned}$$

Thus we have that $\sum |a_n|$ diverges since it yields a harmonic series, and since all terms of a_n are greater than or equal to 0, we have that the series $\sum |a_n| = \sum a_n$. Thus $\sum a_n$ also diverges.

(b) If $\sum |a_n|$ diverges, then $\sum |a_n|$ is conditionally convergent.

This is a false statement. Refer to the previous problem's counterexample: Consider the sequence $a_n = (0, \frac{1}{2}, 0, \frac{1}{4}, 0, \dots)$. we note that all of the terms are greater than or equal to 0. We also note that the sequence can be defined piecewise as follows:

$$a_n := \begin{cases} 0, & n \text{ is odd} \\ \frac{1}{n}, & n \text{ is even} \end{cases}$$

for all $n \in \mathbb{N}$. By this definition, we have

$$\begin{aligned} \sum_{n=1}^{\infty} |a_n| &= \sum_{n=1}^{\infty} a_n \\ &= 0 + \frac{1}{2} + 0 + \frac{1}{4} + 0 + \frac{1}{6} + \dots \\ &= \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots \\ &= \sum_{n=1}^{\infty} \frac{1}{2n} \\ &= \infty \end{aligned}$$

Thus we have that $\sum |a_n|$ diverges since it yields a harmonic series, and since all terms of a_n are greater than or equal to 0, we have that the series $\sum |a_n| = \sum a_n$. Thus $\sum a_n$ also diverges.

(c) If $\sum |a_n|$ diverges, then $\sum a_n$ diverges.

This is a false statement. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. Then, we have that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$, but $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$, since this is a harmonic series. Thus we have that $\sum |a_n|$ diverges since it yields the harmonic series, but $\sum a_n$ converges to $\ln 2$, hence $\sum a_n$ is conditionally convergent.

(d) If $\sum |a_n|$ converges, then $\sum a_n$ is absolutely convergent.

This is true since it is the definition of *Absolute Convergence*.

(e) If $a_n \leq b_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ is absolutely convergent, then $\sum a_n$ converges.

This is true by the *Comparison Test*.

(f) If $\sum a_n$ is absolutely convergent, then $\sum a_n^2$ is absolutely convergent.

This is a true statement.

Proof. Let $\sum a_n$ be an absolutely convergent series. Then $\lim_{n\to\infty} a_n = 0$ by the *nth Term Test*. Thus, we know that $\exists\ N \in \mathbb{N}$ s.t. $0 < a_n < 1$, $\forall\ n \geq N$, and thus $0 < a_n^2 < a_n < 1$, $\forall\ n \geq N$, and thus by the *Comparison Test*, the series $\sum a_n^2$ is also absolutely convergent. ■

(g) If $\lim a_n = 0$, then $\sum(-1)^n a_n$ converges.

This is a false statement. Consider the example given in Problem 3 of Section 9.3:

Let $\sum a_n$ be the series defined as

$$\left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} + \cdots + \frac{1}{n} - \frac{1}{2n} + \dots\right)$$

Let A_n be the partial sums of the series $\sum a_n$. Then since a_n is an alternating sequence that converges to 0 and isn't decreasing, we have

$$A_{2n} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{2n}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{2n}\right)$$

As A_{2n} diverges, we have that A_n diverges and thus the series $\sum a_n$ is divergent.

(h) If $\lim a_n = 0$ and $a_n \geq 0$ for all $n \in \mathbb{N}$, then $\sum(-1)^n a_n$ converges.

This is a false statement. Consider the sequence $a_n = (1, 0, \frac{1}{3}, 0, \frac{1}{5}, 0, \dots)$. we note that all of the terms are greater than or equal to 0. We also note that the sequence can be defined piecewise as follows:

$$a_n := \begin{cases} 0, & n \text{ is even} \\ \frac{1}{n}, & n \text{ is odd} \end{cases}$$

for all $n \in \mathbb{N}$. By this definition, we have that $\lim a_n = 0$.

However, notice

$$\begin{aligned}
\sum_{n=1}^{\infty}(-1)^n a_n &= -1 + 0 - \frac{1}{3} + 0 - \frac{1}{5} + 0 - \frac{1}{7} + 0 - + - + \dots \\
&= -1 - \frac{1}{3} - \frac{1}{5} - \frac{1}{7} - \dots \\
&= -\sum_{n=1}^{\infty} \frac{1}{2n-1} \\
&= -\infty
\end{aligned}$$

Yields a negative harmonic series, which diverges. Hence $\sum(-1)^n a_n$ diverges.

(i) If $\lim a_n = 0$ and $\sum(-1)^n a_n$ converges, then a_n is decreasing.

This is a false statement. Consider the sequence $a_n = (0, \frac{1}{4}, 0, \frac{1}{16}, 0, \dots)$. we note that all of the terms are greater than or equal to 0. We also note that the sequence can be defined piecewise as follows:

$$a_n := \begin{cases} 0, & n \text{ is odd} \\ \frac{1}{n^2}, & n \text{ is even} \end{cases}$$

for all $n \in \mathbb{N}$. By this definition, we have that $\lim a_n = 0$.

However, notice

$$\begin{aligned} \sum_{n=1}^{\infty}(-1)^n a_n &= -0 + \frac{1}{4} - 0 + \frac{1}{16} - 0 + \frac{1}{36} - 0 + - + - \dots \\ &= \frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \dots \\ &= \frac{1}{4}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots\right) \\ &= \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} \\ &= \frac{1}{4} \cdot \frac{\pi^2}{6} \\ &= \frac{\pi^2}{24} \end{aligned}$$

Thus we have that $\sum(-1)^n a_n$ converges. However, the sequence a_n is not a decreasing sequence. Thus $\lim a_n = 0$, and $\sum(-1)^n a_n$ converges, but a_n is not decreasing.

(j) If $a_n \geq 0$ for all n and $\sum a_n$ converges, then $\sum \sin a_n$ converges.

This is a false statement. Consider $a_n = (0, 0, 0, 0, \dots)$. Then we have that $\sum a_n = 0 + 0 + 0 + 0 + \dots = 0$. Thus $\sum a_n$ converges. However, $\sum \sin a_n = \sin(0) + \sin(0) + \sin(0) + \sin(0) + \dots = 1 + 1 + 1 + 1 + \dots = \infty$, and thus $\sum \sin a_n$ diverges.

5. Assume that $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$. Prove that:

(a) $\sum \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

$$\begin{aligned}
\sum \frac{1}{n^2} &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} \dots \\
&= 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots \\
&= \sum \frac{1}{(2n-1)^2} + \frac{1}{4}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots\right) \\
&= \sum \frac{1}{(2n-1)^2} + \frac{1}{4}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\right) \\
&= \sum \frac{1}{(2n-1)^2} + \frac{1}{4}\sum \frac{1}{n^2} \\
&= \sum \frac{1}{(2n-1)^2} + \frac{1}{4} \cdot \frac{\pi^2}{6} \\
&= \sum \frac{1}{(2n-1)^2} + \frac{\pi^2}{24} \\
&\Downarrow \\
\sum \frac{1}{(2n-1)^2} &= \sum \frac{1}{n^2} - \frac{1}{4}\sum \frac{1}{n^2} \\
&= \frac{\pi^2}{6} - \frac{\pi^2}{24} \\
&= \frac{\pi^2}{8}
\end{aligned}$$

(b) $\frac{\pi^2}{24} = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots$

$$\begin{aligned}
\sum \frac{1}{(2n)^2} &= \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots \\
&= \frac{1}{4}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots\right) \\
&= \frac{1}{4}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\right) \\
&= \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} \\
&= \frac{1}{4} \cdot \frac{\pi^2}{6} \\
&= \frac{\pi^2}{24}
\end{aligned}$$

(c) $\dfrac{\pi^2}{12} = 1 - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \dots$

$$\begin{aligned}
\sum \frac{(-1)^{n+1}}{n^2} &= 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \dots \\
&= 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots + \frac{1}{(2n-1)^2} - \frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{6^2} - \dots - \frac{1}{(2n)^2} \\
&= \sum \frac{1}{(2n-1)^2} - \left(\frac{1}{2^1} + \frac{1}{4^2} + \frac{1}{6^2} + \dots + \frac{1}{(2n)^2} \right) \\
&= \sum \frac{1}{(2n-1)^2} - \sum \frac{1}{(2n)^2} \\
&= \frac{\pi^2}{8} - \frac{\pi^2}{24} \\
&= \frac{\pi^2}{12}
\end{aligned}$$

By our previous answers for part (a) and (b).