Real Analysis II Homework 7

Alexander J. Tusa

March 21, 2019

1. **Section 9.1**

7. **i.** If $\sum a_n$ is absolutely convergent and (b_n) is a bounded sequence, show that $\sum a_n b_n$ is absolutely convergent.

Proof. We want to show that $\sum(a_n b_n)$ is also absolutely convergent.
Since (b_n) is bounded, we know that there is a $M > 0$ such that $|b_n| \leq M, \ \forall\ n$.
Then we have that

$$|a_n b_n| = |a_n| \cdot |b_n| \leq M \cdot |a_n|$$

Since $\sum a_n$ is absolutely convergent, $M \cdot \sum |a_n|$ is also convergent. And since $|a_n b_n| \leq M \cdot |a_n|$ we know that $\sum |a_n b_n|$ is also convergent, and therefore $\sum(a_n b_n)$ is absolutely convergent. ∎

ii. Give an example to show that if the convergence of $\sum a_n$ is conditional and (b_n) is a bounded sequence, then $\sum a_n b_n$ may diverge.

Consider the series $\sum a_n = \sum \dfrac{(-1)^n}{n}$ and the bounded sequence $(b_n) = (-1)^n$. We know that $\sum a_n$ is conditionally convergent and (b_n) is bounded by 1. And since $\sum(a_n b_n) = \sum \dfrac{(-1)^n}{n} \cdot (-1)^n = \displaystyle\sum \frac{1}{n}$, we have that the product series is a harmonic series, and thus diverges.

8. Give an example of a convergent series $\sum a_n$ such that $\sum a_n^2$ is not convergent. (Compare this with Exercise 3.7.11)

Consider the series $\sum a_n = \sum \dfrac{(-1)^n}{\sqrt{n}}$, which we know is convergent. But, $\sum (a_n)^2 = \sum \frac{1}{n}$, which is a harmonic series, and thus diverges.

9. If (a_n) is a decreasing sequence of strictly positive numbers and if $\sum a_n$ is convergent, show that $\lim(na_n) = 0$.

Proof. Let (a_n) be a sequence such that $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, and let $\sum a_n$ be convergent, and let $s_n = a_1 + a_2 + \cdots + a_n$.

Let $\varepsilon > 0$ be given. Since $\sum a_n$ is convergent, we know that $\lim\limits_{n\to\infty} s_n = 0$ by the *nth-Term Test*. By the *Cauchy Criterion for Series*, $\exists\ M(\varepsilon) \in \mathbb{N}$ s.t. if $m > n \geq M(\varepsilon)$, then $|s_m - s_n| = |a_{n+1} + a_{n+2} + \cdots + a_m| < \varepsilon$. Now, since (a_n) is a decreasing sequence, we have

$$\begin{aligned} s_{2n} - s_n &= (a_1 + \cdots + a_{2n}) - (a_1 + \cdots + a_n) \\ &= a_{n+1} + \cdots + a_{2n} \\ &\geq a_{2n} + \cdots + a_{2n} \\ &= n \cdot a_{2n} \\ &> 0 \end{aligned}$$

Additionally, we note that

$$\begin{aligned} s_{2n-1} - s_n &= (a_1 + \cdots + a_{2n-1}) - (a_1 + \cdots + a_n) \\ &= a_{n+1} + \cdots + a_{2n-1} \\ &\geq a_{2n-1} + \cdots + a_{2n-1} \\ &= (n-1) \cdot a_{2n} \\ &> 0 \end{aligned}$$

Notice that if we let $n > M(\varepsilon)$, then $|s_{2n} - s_n| < \varepsilon$ and $|s_{2n-1} - s_n| < \varepsilon$.

Choose $\delta = 2M(\varepsilon)$ and for $n > \delta$, we get $n \cdot a_{2n} \cdot (n-1) \cdot a_{2n} = na_n < \varepsilon$, and thus by the definition of a limit, we have that

$$\lim_{n\to\infty} na_n = 0$$

■

10. Give an example of a divergent series $\sum a_n$ with (a_n) decreasing and such that $\lim(na_n) = 0$.

Consider the series $\sum a_n = \sum \frac{1}{n \ln n}$. We showed on the previous homework that this series diverges. And the sequence (a_n) is decreasing since $n \ln n < (n+1)\ln(n+1)$. Thus, we have

$$\lim_{n\to\infty} na_n = \lim_{n\to\infty} n \cdot \frac{1}{n \ln n} = \lim_{n\to\infty} \frac{1}{\ln n} = 0$$

11. If (a_n) is a sequence and if $\lim(n^2 a_n)$ exists in $\mathbb{R}$, show that $\sum a_n$ is absolutely convergent.

Proof. Since $\lim na_n^2$ exists, we know that the sequence $(n^2 a_n)$ is bounded. Thus we know that there exists $M > 0$ such that

$$|n^2 a_n| \leq M \implies |a_n| \leq \frac{M}{n^2},\ \forall\ n$$

Since we know that $\sum \frac{M}{n^2}$ is convergent, by the *Comparison Test*, we know that $\sum |a_n|$ is also convergent, which yields that $\sum a_n$ is absolutely convergent. ■

12. Let $a > 0$. Show that the series $\sum(1 + a^n)^{-1}$ is divergent if $0 < a \leq 1$ and is convergent if $a > 1$.

Proof. Let $a > 0$. We want to show that the series $\displaystyle\sum a_n = \sum \frac{1}{1 + a^n}$ is divergent when $0 < a \leq 1$ and is convergent when $a > 1$.

Case 1 ($0 < a < 1$): We notice that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{1 + a^n} = \frac{1}{1 + \lim\limits_{n\to\infty} a^n} = \frac{1}{1 + 0} = 1 \neq 0$$

And thus by the *nth Term Test*, we have that $\sum a_n$ is divergent when $0 < a < 1$.

Case 2 ($a = 1$): We notice that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{1 + 1^n} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2} \neq 0$$

and thus by the *nth Term Test*, the sum $\sum a_n$ is divergent when $a = 1$.

Case 3 ($a > 1$): We notice that

$$\frac{1}{1 + a^n} < \frac{1}{a^n} = \left(\frac{1}{a}\right)^n$$

which yields a geometric series, and since $a > 1 \implies \left(\frac{1}{a}\right) < 1$, by the *Geometric Series Test*, the series $\sum \left(\frac{1}{a}\right)^n$ converges, and by the *Comparison Test*, the series $\sum \frac{1}{1+a^n}$ must also converge.

■

13. **i.** Does the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}}\right)$ converge?

$$\begin{aligned}
\sum_{n=1}^{\infty} \left(\frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}}\right) &= \sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}} \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \\
&= \sum_{n=1}^{\infty} \frac{(n+1) - n}{\sqrt{n} \cdot (\sqrt{n+1} + \sqrt{n})} \\
&= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n} \cdot (\sqrt{n+1} + \sqrt{n})} \\
&= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)} + n}
\end{aligned}$$

We notice that $\dfrac{1}{\sqrt{n(n+1)}+n} \approx \dfrac{1}{2n}$ and that

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n(n+1)}+n}}{\frac{1}{2n}} &= \lim_{n\to\infty} \frac{2n}{\sqrt{n(n+1)}+n} \\
&= \lim_{n\to\infty} \frac{2n}{\sqrt{n(n+1)}+n} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{2}{\sqrt{1(1+\frac{1}{n})}+1} \\
&= \frac{2}{\sqrt{1(1+0)}+1} \\
&= 1
\end{aligned}$$

Thus we know that since $\sum \frac{1}{n}$ is a harmonic series, it diverges, and thus by the *Comparison Test*, we have that $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}}$ diverges.

ii. Does the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\sqrt{n+1}-\sqrt{n}}{n}\right)$ converge?

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n} &= \sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n} \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \\
&= \sum_{n=1}^{\infty} \frac{(n+1)-n}{n\cdot(\sqrt{n+1}+\sqrt{n})} \\
&= \sum_{n=1}^{\infty} \frac{1}{n\cdot(\sqrt{n+1}+\sqrt{n})}
\end{aligned}$$

We notice that

$$\frac{1}{n\cdot(\sqrt{n+1}+\sqrt{n})} \leq \frac{1}{n\cdot\sqrt{n}} = \frac{1}{n^{\frac{3}{2}}}$$

And since we know that $\displaystyle\sum \frac{1}{n^{\frac{3}{2}}}$ is a convergent p-series, we have that by the *Comparison Test*, the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$ must also be convergent.

2. (a) If $a_n \geq 0$ for all $n \in \mathbb{N}$ and $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, prove that $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n^p}$ converges for all $p \geq 0$.

Proof. We notice that the series $\sum \frac{a_n}{n^p}$ looks like $\sum \frac{1}{n^p}$, which is a convergent p-series when $p > 1$. So, by the *Limit Comparison Test*, we have for $p > 1$:

$$\lim_{n\to\infty} \frac{\frac{a_n}{n^p}}{\frac{1}{n^p}} = \lim_{n\to\infty} \frac{a_n n^p}{n^p} = \lim_{n\to\infty} a_n = 0$$

Since by the *nth Term Test*, since $\sum a_n$ converges, $\lim_{n\to\infty} a_n = 0$. And since $\sum \frac{1}{n^p}$ is a convergent p series, by the *Limit Comparison Test*, the series $\sum_{n=1}^{\infty} \frac{a_n}{n^p}$ is convergent when $p > 1$.

As for the case in which $0 \leq p \leq 1$, consider the following subcases:

Case 4 ($p = 0$):

$$\sum_{n=1}^{\infty} \frac{a_n}{n^p} = \sum_{n=1}^{\infty} \frac{a_n}{n^0} = \sum_{n=1}^{\infty} \frac{a_n}{1} = \sum_{n=1}^{\infty} a_n$$

Which since we assumed that $\sum a_n$ converges, we have that when $p = 0$, $\sum \frac{a_n}{n^p}$ converges.

Case 5 ($p = 1$): We notice that $\frac{a_n}{n} \leq \frac{na_n}{n}$, and thus we have that by the *Comparison Test*, $\sum \frac{na_n}{n} = \sum a_n$, which we know converges, and thus by the *Comparison Test*, $\sum \frac{a_n}{n}$ must also converge.

Case 6 ($0 < p < 1$): We notice that $0 \leq \frac{a_n}{n^p} < \frac{a_n \cdot \sqrt{n^p}}{\sqrt{n^p}}$. So, by the *Comparison Test*, we have

$$\sum_{n=1}^{\infty} \frac{a_n \cdot \sqrt{n^p}}{\sqrt{n^p}} = \sum_{n=1}^{\infty} a_n$$

And since $\sum a_n$ converges, we have that by the *Comparison Test*, $\sum \frac{a_n}{n^p}$ must also converge.

Therefore $\forall\ p \geq 0$, if $\sum a_n$ converges, then $\sum \frac{a_n}{n^p}$ converges.

Alternatively, we notice that $p \geq 0$ implies that $n^p \geq 1$. This yields that $0 \leq \frac{1}{n^p} \leq 1$. Then, since $\sum a_n$ converges, by the *Comparison Test*, we have that $\sum \frac{a_n}{n^p}$ must also converge. ■

(b) Let a_k be a sequence of nonnegative real numbers. Prove: If $\sum a_n$ converges, then its sequence of partial sums is bounded.

Proof. Refer to *Theorem 3.7.3*:

Theorem. Let (x_n) be a sequence of nonnegative real numbers. Then the series $\sum x_n$ converges if and only if the sequence $S = (s_k)$ of partial sums is bounded. In this case,

$$\sum_{n=1}^{\infty} x_n = \lim(x_n) = \sup\{s_k : k \in \mathbb{N}\}$$

$\blacksquare$

(c) Give an example of a series that diverges and whose sequence of partial sums is bounded.

Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ where $\frac{(-1)^n}{n} \neq 0$. This is an alternating series, whose sequence of partial sums is

$$(s_k) = \left(-1, -\frac{1}{2}, -\frac{5}{6}, -\frac{7}{12}, \dots\right)$$

We have that this sequence is bounded below by -1, and is bounded above by 0, however this series diverges since by the *Alternating Series Test*, $\lim_{n\to\infty} \frac{1}{n} = \infty$, and thus the series diverges but the sequence of partial sums is bounded.

(d) Prove: If $\sum |a_n|$ converges and a sequence b_n is bounded, then $\sum a_n b_n$ converges.

Proof. Refer to Problem 7a from Section 9.1:
We want to show that $\sum(a_n b_n)$ is also absolutely convergent.
Since (b_n) is bounded, we know that there is a $M > 0$ such that $|b_n| \leq M$, $\forall\, n$. Then we have that

$$|a_n b_n| = |a_n| \cdot |b_n| \leq M \cdot |a_n|$$

Since $\sum a_n$ is absolutely convergent, $M \cdot \sum |a_n|$ is also convergent. And since $|a_n b_n| \leq M \cdot |a_n|$ we know that $\sum |a_n b_n|$ is also convergent, and therefore $\sum(a_n b_n)$ is absolutely convergent.

By *Theorem 9.1.1*, since $\sum |a_n b_n|$ is absolutely convergent, then the series $\sum a_n b_n$ is also convergent. $\blacksquare$

3. (pr. 18a, Sec. 3.7) Find the positive values of p such that the logarithmic p-series $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$ converges using a) the integral test and b) the Cauchy condensation test.

Since the terms are decreasing, we can use the *Cauchy Condensation Test*.

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} \frac{1}{n(\ln n)^p} \\
\sum_{n=1}^{\infty} 2^n a_{2n} &= \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{2^n(\ln 2^n)^p} \\
&= \sum_{n=1}^{\infty} \frac{1}{(n\ln 2)^p} && (\ln a^b = b\ln a) \\
&= \sum_{n=1}^{\infty} \frac{1}{(\ln 2)^p} \cdot \frac{1}{n^p} \\
&= \frac{1}{(\ln 2)^p} \sum_{n=1}^{\infty} \frac{1}{n^p}
\end{aligned}$$

Now, since $\sum \frac{1}{n^p}$ is a p-series, the only way for the series to converge is if $p > 1$ by the *p-Series Test*, and thus we have that by the *Cauchy Condensation Test* and by the *Comparison Test*, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^p}$ converges if and only if $p > 1$.

So, by the *Integral Test*, let $u = \ln k$ and $du = k\ dk$. Then we have:

$$\int_2^{\infty} \frac{1}{k(\ln k)^p}\ dk = \int_2^{\infty} \frac{1}{u^p}\ du = \int_{\ln 2}^{\infty} u^{-p}\ du = \lim_{b\to\infty} \frac{u^{-p+1}}{1-p}\bigg|_{\ln 2}^{b} = \lim_{b\to\infty} \frac{b^{-p+1}}{1-p} - \frac{(\ln 2)^{-p+1}}{1-p}$$

which we have converges only when $p > 1$ since only then will the b term go to 0.

And by the *Cauchy Condensation Test*, we have $\displaystyle\sum_{n=2}^{\infty} \frac{2^n}{2^n(\ln 2^n)^p} = \sum_{n=2}^{\infty} \frac{1}{(\ln 2^n)^p} = \sum_{n=2}^{\infty} \frac{1}{(n\ln 2)^p} = \frac{1}{(\ln 2)^p}\sum_{n=2}^{\infty} \frac{1}{n^p}$, and by the definition of a p-series this converges when $p > 1$.

4. Give an example of two series $\sum a_n$ and $\sum b_n$ that converge but $\sum a_n b_n$ diverges. (Similar to pr. 8, Sec. 9.1)

Consider the series $\sum a_n = \sum \frac{(-1)^n}{\sqrt{n}}$ and $\sum a_n = \sum \frac{(-1)^{n+1}}{\sqrt{n}}$ which we know is convergent. But, $\sum(a_n b_n) = \sum -\frac{1}{n}$, which is a negative harmonic series, and thus diverges.

5. Prove or justify, if true; Provide a counterexample, if false.

(a) Let a_k and b_k be sequences of positive real numbers. If $\sum a_n$ and $\sum b_n$ converge, then $\sum a_n b_n$ converges.

This is a true statement.

Proof. Since $\sum a_n$ converges, by the *nth Term Test*, $\lim a_n = 0$. So there must exists some $N \in \mathbb{N}$ s.t. $\forall\ n \geq N,\ a_n \leq 1$. So, $0 \leq a_n \leq 1$, which yields that $0 \leq a_n b_n \leq b_n\ \forall\ n \geq N$. Thus, since $\sum b_n$ converges, by the *Comparison Test*, we have that $\sum a_n b_n$ converges. ■

(b) Let a_k and b_k be sequences of positive real numbers. If $\sum a_n b_n$ converges, then $\sum a_n$ and $\sum b_n$ converge.

This is a false statement. Consider the series $\sum \frac{1}{n^3}$ and $\sum n$. Then we have that $\sum \frac{1}{n^3} \cdot n = \sum \frac{1}{n^2}$, which is a convergent p series with $p = 2 > 1$. However, we have that while $\sum \frac{1}{n^3}$ converges since it is a convergent p-series with $p = 3 > 1$, but $\sum n$ diverges.

(c) Let a_k and b_k be sequences of positive real numbers. If $\sum a_n$ and $\sum b_n$ converge, then $\sum \sqrt{(a_n)^2 + (b_n)^2}$ converges.

This is a true statement.

Proof. Let $\sum a_n$ and $\sum b_n$ be convergent series, and let a_n and b_n be positive sequence of real numbers. Then by *Theorem 3.7.3*, we know that the sequences of partial sums (s_k) and (t_k) of $\sum a_n$ and $\sum b_n$, respectively, must be bounded. And since the sequences of partial sums converges, we know that by the *Cauchy Convergence Criterion*, (s_k) and (t_k) are Cauchy sequences. Additionally, by *Theorem 3.2.3*, since both (s_k) and (t_k) converge, then $(s_k) + (t_k)$ must also converge. This yields that $\forall\ \varepsilon > 0,\ \exists\ K(\varepsilon) \in \mathbb{N}$ s.t. $\forall\ n > m \geq K(\varepsilon)$, by the *Triangle Inequality*, we have

$$\sqrt{(s_n - s_m)^2 + (t_n - t_m)^2} \leq \sqrt{(s_n - s_m)^2} + \sqrt{(t_n - t_m)^2} = |s_n - s_m| + |t_n - t_m| < \varepsilon$$

And thus since this is the definition of the *Cauchy Criterion for Series*, we have that this is equal to the series $\sum \sqrt{(a_n)^2 + (b_n)^2}$. Therefore the series $\sum \sqrt{(a_n)^2 + (b_n)^2}$ converges. ■

(d) Let a_k and b_k be sequences of positive real numbers. If $\sum \sqrt{(a_n)^2 + (b_n)^2}$ converges, then $\sum a_n$ and $\sum b_n$ converge.

This is a true statement.

Proof. By the *Comparison Test*, since $0 \leq a_n^2 \leq a_n^2 + b_n^2$, we have that $0 \leq a_n \leq \sqrt{a_n^2 + b_n^2}$, and thus $\sum a_n$ converges. Similarly, by the *Comparison Test*, we have that since $0 \leq b_n^2 \leq a_n^2 + b_n^2$, we have that $0 \leq b_n \leq \sqrt{a_n^2 + b_n^2}$, and thus $\sum b_n$ converges. ■

(e) If $\sum a_n$ converges and $0 \leq b_n \leq a_n$, then $\sum b_n$ converges.

This is true since it is the *Comparison Test*.

(f) If $\lim\limits_{n\to\infty} a_n = 0,\ a_n \geq 0$ and $\sqrt{a_{n+1}} \leq a_n$ for all $n \in \mathbb{N}$, then $\sum a_n$ converges.

Proof. Since $\lim a_n = 0$, we know that $\exists\ N \in \mathbb{N}$ s.t. $\forall\ n \geq N,\ |a_n| \leq \frac{1}{2}$. (That is, let $\varepsilon = \frac{1}{2}$). Also, since $\sqrt{a_{n+1}} \leq a_n$, we know that $a_{n+1} \leq a_N^2 \leq \frac{1}{4}$. So $a_{n+1} \leq \frac{1}{4}$. So, we have

$$a_{n+2} \leq a_{n+1}^2 \leq \frac{1}{16} = \frac{1}{4^2}$$
$$\cdots$$
$$a_{n+3} \leq \frac{1}{4^4}$$

So, we have that

$$|a_n| \leq \left(\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots + \frac{1}{4^{n-N}}\right)$$

since $a_n = a_{N(n-N)}$. So, we have that $0 \leq |a_n| \leq \frac{1}{4}$, which is a convergent geometric series. Therefore by the *Comparison Test*, we have that $\sum a_n$ is convergent. ■