# Real Analysis II Homework 6

Alexander J. Tusa

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**1. Section 3.7**

**6.** **i.** Calculate the value of $\displaystyle\sum_{n=2}^{\infty}\left(\frac{2}{7}\right)^n$. (Note the series starts at $n = 2$)

$$\begin{aligned}
\sum_{n=2}^{\infty}\left(\frac{2}{7}\right)^n &= \sum_{n=1}^{\infty}\left(\frac{2}{7}\right)^{n+2} \\
&= \sum_{n=1}^{\infty}\left(\frac{2}{7}\right)^n - \frac{2}{7} - 1 \\
&= \frac{1}{1-\frac{2}{7}} - \frac{2}{7} - 1 \\
&= \frac{1}{\frac{5}{7}} - \frac{2}{7} - 1 \\
&= \frac{7}{5} - \frac{2}{7} - 1 \\
&= \frac{4}{35}
\end{aligned}$$

**ii.** Calculate the value of $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{2n}$. (Note the series starts at $n = 1$)

$$\begin{aligned}
\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{2n} &= \sum_{n=1}^{\infty}\left(\frac{1}{9}\right)^n \\
&= \sum_{n=1}^{\infty}\left(\frac{1}{9}\right)^n - 1 \\
&= \frac{1}{1-\frac{1}{9}} - 1 \\
&= \frac{1}{\frac{8}{9}} - 1 \\
&= \frac{9}{8} - 1 \\
&= \frac{1}{8}
\end{aligned}$$

**7.** Find a formula for the series $\sum_{n=1}^{\infty} r^{2n}$ when $|r| < 1$.

Let $S_n = r^2 + r^4 + r^6 + \cdots + r^{2(n-1)} + r^{2n}$. Then, we notice that $S_n = (r^2)^1 + (r^2)^2 + (r^2)^3 + \cdots + (r^2)^{n-1} + (r^2)^n$ which in turn yields $S_n = r^2[1 + r^2 + (r^2)^2 + (r^2)^3 + \cdots + (r^2)^{n-1}]$ yielding $S_n = r^2 \cdot \frac{1-(r^2)^n}{1-r^2} = \frac{r^2}{1-r^2} \cdot (1 - (r^2)^n)$. Since $|r| < 1$, we know that $|r^2| < 1$, and thus $\lim_{n\to\infty} (r^2)^n = 0$. So $\lim S_n = \frac{r^2}{1-r^2} \cdot (1 - 0) = \frac{r^2}{1-r^2}$. Thus $\sum_{n=1}^{\infty} r^{2n}$ converges and is equal to $\frac{r^2}{1-r^2}$.

**9.** **i.** Show that the series $\sum_{n=1}^{\infty} \cos n$ is divergent.

We note that if the series converges, then $\lim_{n\to\infty} \cos n = 0$. Consider the subsequence $n_k = 2k\pi$, for some $k \in \mathbb{N}$. Then $\lim_{k\to\infty} \cos(2k\pi) = 1$. And for the subsequence $n_k = \frac{\pi}{2} + 2k\pi$, for some $k \in \mathbb{N}$, then we have that $\lim_{k\to\infty} \cos\left(\frac{\pi}{2} + 2k\pi\right) = 0$. And since $0 \neq 1$, we can conclude that $\cos n$ does not converge and that $\lim_{n\to\infty} \cos n \neq 0$. Therefore, by the *nth Term Test*, $\sum_{n=1}^{\infty} \cos n$ diverges.

**ii.** Show that the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is convergent.

We note that $\left|\frac{\cos n}{n^2}\right| \leq \frac{1}{n^2}$, and since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent $p$-series, we have that $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is also convergent.

**11.** If $\sum a_n$ with $a_n > 0$ is convergent, then is $\sum a_n^2$ always convergent? Either prove it or give a counterexample.

*Proof.* Since $\sum a_n$ is convergent, we know that by the *nth Term Test*, $\lim a_n = 0$. This yields that for $\varepsilon = 1 > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\begin{aligned} |a_n - 0| &< \varepsilon \\ |a_n| &< \varepsilon \\ |a_n| &< 1 \end{aligned}$$

$$0 < a_n < 1$$

since $a_n > 0$, $\forall n$. Since we know that $x^2 < x$ for $0 < x < 1$, we have that $0 < a_n^2 < a_n < 1$, $\forall n \geq N$. Thus, by the *Comparison Test*, since $0 \leq a_n^2 \leq a_n$ and since $\sum a_n$ converges, we have that $\sum a_n^2$ converges. ∎

**12.** If $\sum a_n$ with $a_n > 0$ is convergent, then is $\sum \sqrt{a_n}$ always convergent? Either prove it or give a counterexample.

This is a false statement. Consider $\sum a_n = \sum \frac{1}{n^2}$. Then we have that $\sum \sqrt{a_n} = \sum \frac{1}{n}$, which is a harmonic series, which we know diverges. Thus if $\sum a_n$ converges, then $\sum \sqrt{a_n}$ does not necessarily converge.

**13.** If $\sum a_n$ with $a_n > 0$ is convergent, then is $\sum \sqrt{a_n a_{n+1}}$ always convergent? Either prove it or give a counterexample.

*Proof.* We first notice that

$$\frac{a_n + a_{n+1}}{2} > \sqrt{a_n a_{n+1}}$$

So, we then have

$$\frac{a_1}{2} + \sum_{n=1}^{\infty} \left( \frac{a_n + a_{n+1}}{2} \right) = \frac{a_1}{2} + \frac{a_1}{2} + \frac{a_2}{2} + \frac{a_2}{2} + \frac{a_3}{2} + \frac{a_3}{2} + \cdots = a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$$

thus, since $\frac{a_1}{2} + \sum_{n=1}^{\infty} \left( \frac{a_n + a_{n+1}}{2} \right) = \sum_{n=1}^{\infty} a_n$, and since we're given that $\sum a_n$ converges, then $\sum_{n=1}^{\infty} \left( \frac{a_n + a_{n+1}}{2} \right)$ converges. Thus by the *Comparison Test*, we have that

$$a_n > \left( \frac{a_n + a_{n+1}}{2} \right) > \sqrt{a_n a_{n+1}}$$

yields that $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ also converges. ∎

**16.** Use the *Cauchy Condensation Test* to discuss the $p$-series $\sum_{n=1}^{\infty} (1/n^p)$ for $p > 0$.

By the *Cauchy Condensation Test*, we must show that the series $\sum_{n=0}^{\infty} 2^n \cdot a(2^n)$ converges. So,

$$\begin{aligned} \sum_{n=0}^{\infty} 2^n a(2^n) &= \sum_{n=0}^{\infty} \left[ 2^n \frac{1}{(2^n)^p} \right] \\ &= \sum_{n=0}^{\infty} 2^n \left[ \frac{1}{2^{np}} \right] \\ &= \sum_{n=0}^{\infty} 2^n (2^{-np}) \\ &= \sum_{n=0}^{\infty} 2^{n(1-p)} \\ &= \sum_{n=0}^{\infty} (2^{1-p})^n \end{aligned}$$

We notice that this is now a geometric series, with $|r| = |2^{1-p}|$. Then we note that if $|2^{1-p}| \leq 1$, the series converges, and otherwise the series diverges.

**17.** Use the *Cauchy Condensation Test* to establish the divergence of the series:

**i.** $\sum \frac{1}{n \ln n}$

By the *Cauchy Condensation Test*, we have

$$\begin{aligned}\sum \frac{1}{n \ln n} &= \sum 2^n \cdot \frac{1}{2^n \cdot \ln 2^n} \\ &= \sum \frac{1}{\ln 2^n} \\ &= \sum \frac{1}{n \ln 2} \\ &= \frac{1}{\ln 2} \sum \frac{1}{n}\end{aligned}$$

Since $\sum \frac{1}{n}$ is a harmonic series, we know that the series diverges. Thus we have that $\sum 2^n \cdot \frac{1}{2^n \cdot \ln 2^n}$ diverges as well. Thus by the *Cauchy Condensation Test*, we have that $\sum \frac{1}{n \ln n}$ diverges also.

**ii.** $\sum \frac{1}{n(\ln n)(\ln \ln n)}$

By the *Cauchy Condensation Test*, we have

$$\begin{aligned}\sum \frac{1}{n(\ln n)(\ln \ln n)} &= \sum 2^n \frac{1}{2^n \cdot (ln 2^n) \cdot (\ln(\ln 2^n))} \\ &= \sum \frac{1}{\ln 2^n \cdot \ln \ln 2^n} \\ &= \sum \frac{1}{(n \ln 2) \cdot (\ln(n \ln 2))} \\ &= \sum \frac{1}{(n \ln 2) \cdot (\ln n + \ln \ln 2)} \\ &> \sum \frac{1}{n \cdot (\ln 2 + \ln \ln 2} & (\ln 2 < 1) \\ &> \sum \frac{1}{n \cdot \ln n}\end{aligned}$$

Since we showed in part (a) that $\sum 2^n \cdot \frac{1}{n \ln n}$ diverges, and thus by the *Comparison Test*, since $0 \leq \frac{1}{n(\ln n)(\ln \ln n)} < \frac{1}{n \ln n}$, since $\sum \frac{1}{n \ln n}$ diverges, then so does $\sum \frac{1}{(\ln 2^n)(\ln \ln 2^n)}$.

**2.** Use the tests in section 3.7 to test the given series for convergence or divergence. State clearly which test is used. Also, for parts a-f, write out the first three terms of each series.

**(a)** $\displaystyle\sum_{n=1}^{\infty} \frac{2n+5}{3n^2+2n-1}$

First three terms:

$$\sum_{n=1}^{\infty} \frac{2n+5}{3n^2+2n-1} = \frac{7}{4} + \frac{3}{5} + \frac{11}{32} + \dots$$

Note that $2n < 2n+5$ and that $3n^2+2n-1 < 4n^2 \implies \frac{1}{3n^2+2n-1} > \frac{1}{4n^2}$, and thus $\frac{2n}{4n^2} = \frac{1}{2n} < \frac{2n+5}{3n^2+2n-1}$. Since $\sum \frac{1}{2n} = \frac{1}{2}\sum \frac{1}{n}$ yields a half times the harmonic series, which we know is divergent, we have that by the *Comparison Test*, $\sum \frac{2n+5}{3n^2+2n-1}$ also diverges.

**(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n2^n}$

The first three terms:

$$\sum_{n=1}^{\infty} \frac{n-1}{n2^n} = 0 + \frac{1}{8} + \frac{1}{12} + \dots$$

We note that $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n2^n}$ looks similar to $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$, which is a geometric series. We also note that the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges since $\left|\frac{1}{2}\right| < 1$. Then, by the *Limit Comparison Test*, we have $\displaystyle\lim_{n\to\infty} \left(\frac{\frac{n-1}{n2^n}}{\left(\frac{1}{2}\right)^n}\right) = 1 \neq 0$, and thus since $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges, then $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n2^n}$ must also converge.

**(c)** $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+\pi}{2+\sqrt[5]{n^8}}$

The first three terms of this series are:

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}+\pi}{2+\sqrt[5]{n^8}} = \frac{\pi+1}{3} + \frac{\pi+\sqrt{2}}{2+2\sqrt[5]{8}} + \frac{\pi+\sqrt{3}}{2+3\sqrt[5]{27}} + \dots$$

We note that $\sqrt{n}+\pi < \sqrt{n}$ and that $2+\sqrt[5]{n^8} > n^{\frac{7}{4}} \implies \frac{1}{2+\sqrt[5]{n^8}} < \frac{1}{n^{\frac{7}{4}}}$. Thus we have that

$$\frac{\sqrt{n}+\pi}{2+\sqrt[5]{n^8}} < \frac{\sqrt{n}}{n^{\frac{7}{4}}} = \frac{1}{n^{\frac{5}{4}}}$$

We note that $\sum \frac{1}{n^{\frac{5}{4}}}$ converges because it is a $p$-series where $p > 1$. Thus by the *Comparison Test*, we have that $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+\pi}{2+\sqrt[5]{n^8}}$ must also converge.

**(d)** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\ln n}}$

The first three terms are:

$$\sum_{n=1}^{\infty} \frac{1}{n^{\ln n}} = 1 + \frac{1}{e^{(\ln 2)^2}} + \frac{1}{e^{(\ln 3)^2}} + \dots$$

We note that $\frac{1}{n^{\ln n}}$ looks like $\frac{1}{n^{\ln 3}}$ and that $0 < \dfrac{1}{n^{\ln n}} \leq \dfrac{1}{n^{\ln 3}}$. By the *Comparison Test*, since $\dfrac{1}{n^{\ln 3}}$ is a convergent $p$-series since $\ln 3 \geq 1$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\ln n}}$ must also be convergent.

**(e)** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

The first three terms of the series are:

$$\sum_{n=2}^{\infty} \frac{1}{\ln n} = \frac{1}{\ln(2)} + \frac{1}{\ln(3)} + \frac{1}{\ln(4)} + \dots$$

We note that $n > \ln n \implies \frac{1}{n} < \frac{1}{\ln n}$. Then by the *Comparison Test*, since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ is the Harmonic series, we know that it diverges, and thus by the *Comparison Test*, since $\sum \frac{1}{n} \leq \frac{1}{\ln n}$, we have that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ must also diverge.

**(f)** $\displaystyle\sum_{n=2}^{\infty} \frac{n + \ln n}{n^2 + 1}$

The first three terms of the series are:

$$\sum_{n=2}^{\infty} \frac{n + \ln n}{n^2 + 1} = \frac{\ln(2) + 2}{5} + \frac{\ln(3) + 3}{10} + \frac{\ln(4) + 4}{17} + \dots$$

We notice that $\dfrac{n + \ln n}{n^2 + 1}$ looks like $\dfrac{1}{n}$. So, by the *Limit Comparison Test*, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{\frac{n+\ln n}{n^2+1}}{\frac{1}{n}} &= \lim_{n\to\infty} \frac{n^2 + n\ln n}{n^2 + 1} \\ &= \lim_{n\to\infty} \frac{1 + \frac{\ln n}{n}}{1 + \frac{1}{n^2}} \\ &= \frac{1+0}{1+0} \\ &= \frac{1}{1} \\ &= 1 \end{aligned}$$

Thus by the *Limit Comparison Test*, since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, we have that $\sum_{n=2}^{\infty} \frac{n + \ln n}{n^2 + 1}$ diverges.

**(g)** $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$

We note that $n < n + 1$ and that $n^2 > \frac{n^2}{2} \implies \frac{1}{n^2} < \frac{2}{n^2}$ and gives us that

$$\frac{2n}{n^2} = \frac{2}{n} < \frac{n+1}{n^2}$$

Since $2\sum_{n=1}^{\infty} \frac{1}{n}$ is a harmonic series, it is divergent. Thus by the *Comparison Test*, we have that $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$ is also divergent.

**(h)** $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$

By the *Cauchy Ratio Test*, we have the following:

$$\lim_{n\to\infty} \left| \frac{\frac{1}{\ln(n+1)^{n+1}}}{\frac{1}{\ln(n)^n}} \right| = \lim_{n\to\infty} \left| \frac{\ln(n)^n}{\ln(n+1)^{n+1}} \right| = 0$$

Thus by the *Cauchy Ratio Test*, since $L = 0 < 1$, we have that $\sum_{n=1}^{\infty} \frac{1}{\ln(n)^n}$ is a convergent series.

**(i)** $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

Notice that $\frac{\ln n}{n^2} < \frac{1}{n^2}$. So, we must note that $\sum_{n=2}^{\infty} \frac{1}{n^2}$ is a convergent series since it is a $p$-series with $p = 2$ and $p > 0$. Thus by the *Comparison Test*, we have that the series $\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2}$ is a convergent series.

**(j)** $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

We notice that $\frac{\ln n}{n}$ looks like $\frac{1}{n}$. So, by the *Limit Comparison Test*, we have

$$\lim_{n\to\infty} \frac{\frac{\ln n}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \ln n = \infty$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series, we know that it diverges, and thus by the *Limit Comparison Test*, we have that $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges as well.

**(k)** $\sum_{n=3}^{\infty} \frac{\sqrt{n}}{\sqrt{n^3}+1}$

Notice that $\sqrt{n^3} < 2\sqrt{n^3} \implies \frac{1}{\sqrt{n^3}+1} > \frac{1}{2\sqrt{n^3}}$, which gives us that

$$\frac{\sqrt{n}}{2\sqrt{n^3}} < \frac{\sqrt{n}}{\sqrt{n^3}+1}$$

Since the series $\frac{1}{2}\sum_{n=3}^{\infty} \frac{1}{n}$ is a harmonic series ,we know that it diverges, and thus we know that by the *Comparison Test* we have that the series $\sum_{n=3}^{\infty} \frac{\sqrt{n}}{\sqrt{n^3}+1}$ is also divergent.

**(l)** $\sum_{n=1}^{\infty} \frac{n}{e^n}$

By the *McClaurin Integral Test*, we have

$$\int_1^{\infty} \frac{x}{e^x}\ dx = \int_1^{\infty} xe^{-x}\ dx$$

By *Integration by Parts*, let $u = x,\ dv = e^{-x}dx,\ du = dx,\ v = -e^{-x}$. Then we have

$$\begin{aligned}
\int_1^{\infty} xe^{-x}\ dx &= -xe^{-x} + \int_1^{\infty} e^{-x}\ dx \\
&= -xe^{-x} - e^{-x}\Big|_1^{\infty} \\
&= \frac{-\infty}{e^{\infty}} - \frac{1}{e^{\infty}} + \frac{1}{e} + \frac{1}{e} \\
&= \frac{2}{e}
\end{aligned}$$

Thus, since $\int_1^{\infty} \frac{x}{e^x}\ dx$ converges to $\frac{2}{e}$, we have that $\sum_{n=1}^{\infty} \frac{n}{e^n}$ converges.

**3.** Write the given expressions as a quotient of two integers.

**(a)** $3 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots$

$$\begin{aligned}
&= 3 + 1 + \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{3^n} \\
&= 3 + 1 + \frac{1}{2} + \frac{\frac{1}{3}}{1 - \frac{1}{3}} \\
&= 3 + 1 + \frac{1}{2} + \frac{\frac{1}{3}}{\frac{2}{3}} \\
&= 3 + 1 + \frac{1}{2} + \frac{3}{6} \\
&= 3 + 1 + \frac{1}{2} + \frac{1}{2} \\
&= 3 + 1 + 1 \\
&= 5
\end{aligned}$$

**(b)** $3.2\overline{15}$

$$\begin{aligned}
3.2\overline{15} &= 3 + \frac{2}{10} + \left(\frac{15}{1000} + \frac{15}{100000} + \frac{15}{10^7} + \dots\right) \\
&= 3 + \frac{2}{10} + \frac{15}{1000}\left(1 + \frac{1}{10^2} + \frac{1}{10^3} + \dots\right) \\
&= 3 + \frac{2}{10} + \frac{\frac{15}{1000}}{1 - \frac{1}{100}} \\
&= 3 + \frac{2}{10} + \frac{1}{66} \\
&= \frac{1061}{330}
\end{aligned}$$

**4.** **(a)** Let $a_n$ be a sequence of real numbers and $b_n = a_n - a_{n-1}$ for all $n \in \mathbb{N}$. Prove: $\sum b_k$ converges if and only if the sequence $a_n$ converges. In this case, find the sum of $\sum b_k$.

*Proof.* Let $(a_n) \subseteq \mathbb{R}$ and let $b_n := a_n - a_{n-1} \ \forall \ n \in \mathbb{N}$.

(=>) Assume $\sum b_k$ converges. Then by the *Cauchy Criterion for Series*, we have

$$\forall \ \varepsilon > 0, \ \exists \ M(\varepsilon) \in \mathbb{N} \text{ s.t. if } m > n \geq M(\varepsilon) \implies |s_m - s_n| < \varepsilon$$

This yields that

$$\begin{aligned}
|s_m - s_n| &= |(\cancel{a_{n+1}} - a_n) + (\cancel{a_{n+2}} - \cancel{a_{n+1}}) + \dots + (\cancel{a_{m-1}} - \cancel{a_{m-2}}) + (a_m - \cancel{a_{m-1}})| \\
&= |a_m - a_n| \\
&< \varepsilon
\end{aligned}$$

Thus by the definition of a *Cauchy Sequence*, $a_n$ is a Cauchy sequence and thus by the *Cauchy Convergence Criterion*, $a_n$ is a convergent sequence.

(<=) Similarly, suppose $a_n$ is a convergent sequence. Then by the *Cauchy Convergence Criterion*, $a_n$ is a Cauchy sequence. So $\forall\ \varepsilon > 0,\ \exists\ H(\varepsilon) \in \mathbb{N}$ s.t. $\forall\ m > n \geq H(\varepsilon),\ n, m \in \mathbb{N},\ |a_m - a_n| < \varepsilon$. So,

$$\begin{aligned}
|a_m - a_n| &= |(a_{n+1} - a_n) + (a_{n+2} - a_{n+1}) + \cdots + (a_{m-1} - a_{m-2}) + (a_m - a_{m-1})| \\
&= |(s_{m-1} + a_m) - (s_{n-1} + a_n)|, \\
&\text{by the definition of the infinite series generated by } (a_n) \\
&= |s_{m-1} + a_m - s_{n-1} - a_n| \\
&= |s_m - s_n| \\
&< \varepsilon
\end{aligned}$$

Thus by the *Cauchy Criterion for Series*, since $|s_m - s_n| < \varepsilon$ for all $m > n \geq H(\varepsilon), H(\varepsilon), m, n \in \mathbb{N}$, the series $\sum a_n - a_{n-1}$ converges, which implies that the series $\sum b_n$ converges.

■

Thus, we have that $\sum b_n = \lim\limits_{n\to\infty}(b_n) = \lim\limits_{n\to\infty}(a_n - a_{n-1})$, by *Theorem 3.7.3*, and thus by the *nth Term Test*, since $\sum b_n$ converges, we have that $\lim\limits_{n\to\infty}(a_n - a_{n-1}) = 0$. Thus $\sum b_n = 0$.

**(b)** Let $a_n$ be a sequence of real numbers. If $\lim\limits_{n\to\infty} a_n = A$, find the sum of

$$\sum_{n=1}^{\infty}(a_{n+1} - 2a_n + a_{n-1})$$

By the *Cauchy Convergence Criterion for Series*, we have that $\forall\ \varepsilon > 0,\ \exists\ M(\varepsilon) \in \mathbb{N}$ s.t. if $m, n \in \mathbb{N},\ m > n \geq M(\varepsilon) \implies |s_m - s_n| = |a_{n+1} + a_{n+2} + \cdots + a_m| < \varepsilon$. So,

$$\begin{aligned}
|s_m - s_n| = |(&\cancel{a_{n+2}} - \cancel{2}a_{n+1} + a_n) + (\cancel{a_{n+3}} - \cancel{2a_{n+2}} + \cancel{a_{n+1}}) \\
&+ (\cancel{a_{n+4}} - \cancel{2a_{n+3}} + \cancel{a_{n+2}}) + (\cancel{a_{n+5}} - \cancel{2a_{n+4}} + \cancel{a_{n+3}}) \\
&+ \ldots \\
&+ (\cancel{a_m} - \cancel{2a_{m-1}} + \cancel{a_{m-2}}) + (a_{m+1} - \cancel{2}a_m + \cancel{a_{m-1}})| \\
&= |a_n - a_{n+1} + a_{m+1} - a_m| \\
&< \varepsilon
\end{aligned}$$

This yields that the sequence of partial sums of $a_{n+1} - 2a_n + a_{n-1}$ is bounded. Thus by *Theorem 3.7.3*, $\sum\limits_{n=1}^{\infty}(a_{n+1} - 2a_n + a_{n-1}) = \lim\limits_{n\to\infty}(a_{n+1} - 2a_n + a_{n-1}) = A - 2A + A = 0$.

**(c)** If $\sum\limits_{k=1}^{n}(k\ a_k) = \dfrac{n+1}{n+2}$ for $n \in \mathbb{N}$, show that $\sum\limits_{k=1}^{\infty} a_k = \dfrac{3}{4}$.

*Proof.* If $n = 1$, then $\displaystyle\sum_{k=1}^{1} 1 \cdot a_1 = \frac{2}{3}$.

If $n \geq 2$, then

$$\begin{aligned}
\sum_{k=1}^{n} ka_k &= a_1 + a_2 + \cdots + (n-1)a_{n-1} + na_n \\
&= \frac{n+1}{n+2} \\
&\Downarrow \\
na_n &= \frac{n+1}{n+2} - \frac{n}{n+1} \\
&= \frac{1}{(n+1)(n+2)}
\end{aligned}$$

So $a_n = \frac{1}{n(n+1)(n+2)}$, thus we have the following:

$$\begin{aligned}
\sum_{n=2}^{\infty} &= \sum_{n=2}^{\infty} \frac{1}{n(n+1)(n+2)} \\
&= \frac{1}{2} \sum_{n=2}^{\infty} \left[ \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} \right] \\
&= \frac{1}{12} && \text{as we did on the previous homework}
\end{aligned}$$

So $\displaystyle\sum_{n=1}^{\infty} a_n = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}$. ■