Real Analysis II Homework 5

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1. Find the sum of the following series.

(a) (pr. 3a) $\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$

$$\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2} = \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$$

$$\Downarrow$$

$$\begin{aligned} \frac{1}{n^2+3n+2} &= \frac{A}{n+1} + \frac{B}{n+2} \\ 1 &= A(n+2) + B(n+2) \\ 1 &= An + 2A + Bn + 2B \\ 1 &= An + Bn + 2A + 2B \\ 1 &= (A+B)n + (A+B)2 \end{aligned}$$

$$\begin{cases} 0 = A + B \\ 1 = A + B \end{cases} \implies \begin{cases} A = 1 \\ B = -1 \end{cases}$$

$$\begin{aligned} &= \sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+2} \\ &= \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \cancel{\frac{1}{n+1}}\right) + \left(\cancel{\frac{1}{n+1}} - \frac{1}{n+2}\right) \\ &= \lim_{n\to\infty} \frac{1}{2} + \frac{1}{n+2} \\ &= \frac{1}{2} \end{aligned}$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n+1)}$

$$\begin{aligned}
\frac{1}{(3n-2)(3n+1)} &= \frac{A}{3n-2} + \frac{B}{3n+1} \\
1 &= A(3n+1) + B(3n-2) \\
1 &= 3An + 3Bn + A - 2B
\end{aligned}$$

$$\begin{cases} 3A + 3B = 0 \\ 1A - 2B = 1 \end{cases} \implies \begin{cases} A = \frac{1}{3} \\ B = \frac{-1}{3} \end{cases}$$

$$\begin{aligned}
\frac{1}{(3n-2)(3n+1)} &= \frac{1}{9n-6} - \frac{1}{9n+3} \\
\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n+1)} &= \sum_{n=1}^{\infty} \frac{1}{9n-6} - \frac{1}{9n+3} \\
&= \left(\frac{1}{3} - \cancel{\frac{1}{12}}\right) + \left(\cancel{\frac{1}{12}} - \cancel{\frac{1}{21}}\right) + \dots \\
&+ \left(\cancel{\frac{1}{9n-15}} - \cancel{\frac{1}{9n-6}}\right) + \left(\cancel{\frac{1}{9n-6}} - \frac{1}{9n+3}\right) \\
&= \lim_{n\to\infty} \frac{1}{3} - \frac{1}{9n+3} \\
&= \frac{1}{3}
\end{aligned}$$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{e^{n-1}}$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{e^{n-1}} &= \sum_{n=1}^{\infty} \frac{(-1)^n \cdot (-1)^1}{e^n \cdot e^{-1}} \\
&= -\sum_{n=1}^{\infty} \frac{(-1)^n \cdot e}{e^n} \\
&= -\sum_{n=1}^{\infty} (-1)^n \cdot \frac{e}{e^n} \\
&= -\sum_{n=1}^{\infty} (-1)^n \cdot e^{1-n} \\
&= \frac{e}{1+e}
\end{aligned}$$

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{4^{n+1}}{9^{n-1}}$

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{4^{n+1}}{9^{n-1}} &= \sum_{n=2}^{\infty} \frac{4^n \cdot 4^1}{9^n \cdot 9^{-1}} \\
&= \sum_{n=2}^{\infty} \left(\frac{4}{9}\right)^n \cdot 4^1 \cdot 9^1 \\
&= \sum_{n=2}^{\infty} \left(\frac{4}{9}\right)^n \cdot 36 \\
&= \sum_{n=2}^{\infty} \left(\frac{4}{9}\right)^n \cdot 36 - 16 - 36 \\
&= a\left(\frac{1}{1-r}\right) \\
&= 36 \cdot \left(\frac{1}{1-\left(\frac{4}{9}\right)}\right) - 52 \\
&= \frac{36 \cdot 9}{5} - 52 \\
&= \frac{324}{5} - 52 \\
&= \frac{324}{5} - \frac{260}{5} \\
&= \frac{64}{5}
\end{aligned}$$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{5^{n+1} + (-3)^n}{7^{n+2}}$

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{5^{n+1} + (-3)^n}{7^{n+2}} &= \sum_{n=0}^{\infty} \frac{5^{n+1}}{7^{n+2}} + \frac{(-3)^n}{7^{n+2}} \\
&= \sum_{n=0}^{\infty} \frac{5}{49} \cdot \left(\frac{5}{7}\right)^n + \frac{1}{49} \cdot \left(\frac{(-3)}{7}\right)^n \\
&= \sum_{n=0}^{\infty} \frac{5}{49} \cdot \left(\frac{5}{7}\right)^n + \sum_{n=0}^{\infty} \frac{1}{49} \cdot \left(\frac{(-3)}{7}\right)^n \\
&= \frac{5}{49} \cdot \frac{1}{1-\frac{5}{7}} + \frac{1}{49} \cdot \frac{1}{1-\frac{-3}{7}} \\
&= \frac{5}{14} + \frac{1}{70} \\
&= \frac{13}{35}
\end{aligned}$$

(f) $\sum_{n=2}^{\infty} \ln \frac{n^2-1}{n^2}$

$$\begin{aligned}
\sum_{n=2}^{\infty} \ln\left(\frac{n^2-1}{n^2}\right) &= \sum_{n=2}^{\infty} \ln\left(\frac{(n-1)(n+1)}{n^2}\right) \\
&= \sum_{n=2}^{\infty} \ln\left(\frac{\frac{n-1}{n}}{\frac{n}{n+1}}\right) \\
&= \sum_{n=2}^{\infty} \ln\left(\frac{n-1}{n}\right) - \ln\left(\frac{n}{n+1}\right) \\
&= \left(\ln\frac{1}{2} - \cancel{\ln\frac{2}{3}}\right) + \left(\cancel{\ln\frac{2}{3}} - \cancel{\ln\frac{3}{4}}\right) + \\
&\cdots + \left(\cancel{\ln\frac{n-2}{n-1}} - \cancel{\ln\frac{n-1}{n}}\right) + \left(\cancel{\ln\frac{n-1}{n}} - \ln\frac{n}{n+1}\right) \\
&= \lim_{n\to\infty} \ln\left(\frac{1}{2}\right) - \ln\left(\frac{n}{n+1}\right) \\
&= \ln\left(\frac{1}{2}\right) - \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{n+1-n}{(n+1)^2}, && \text{by L'Hospital's Rule} \\
&= \ln\left(\frac{1}{2}\right) - \lim_{n\to\infty} \frac{1}{n(n+1)} \\
&= \ln\left(\frac{1}{2}\right) - 0 \\
&= \ln\left(\frac{1}{2}\right) \\
&\approx -0.693147
\end{aligned}$$

(g) $\displaystyle\sum_{n=2}^{\infty} \ln \frac{n(n+2)}{(n+1)^2}$

$$\begin{aligned}
\sum_{n=2}^{\infty} \ln\left(\frac{n(n+2)}{(n+1)^2}\right) &= \sum_{n=2}^{\infty} \ln\left(\frac{\frac{n}{n+1}}{\frac{n+1}{n+2}}\right) \\
&= \sum_{n=2}^{\infty} \ln\left(\frac{n}{n+1}\right) - \ln\left(\frac{n+1}{n+2}\right) \\
&= \left(\ln\frac{2}{3} - \cancel{\ln\frac{3}{4}}\right) + \left(\cancel{\ln\frac{3}{4}} - \cancel{\ln\frac{4}{5}}\right) + \dots \\
&+ \left(\cancel{\ln\frac{n-1}{n}} - \cancel{\ln\frac{n}{n+1}}\right) + \left(\cancel{\ln\frac{n}{n+1}} - \ln\frac{n+1}{n+2}\right) \\
&= \lim_{n\to\infty} \ln\left(\frac{2}{3}\right) - \ln\left(\frac{n+1}{n+2}\right) \\
&= \ln\left(\frac{2}{3}\right) - \lim_{n\to\infty} \frac{n+2}{n+1} \cdot \frac{1\cdot(n+2)-(n+1)\cdot 1}{(n+2)^2}, \qquad \text{by L'Hospital's Rule} \\
&= \ln\left(\frac{2}{3}\right) - \lim_{n\to\infty} \frac{1}{(n+1)(n+2)} \\
&= \ln\left(\frac{2}{3}\right) - 0 \\
&= \ln\left(\frac{2}{3}\right) \\
&\approx -0.405465
\end{aligned}$$

(h) (pr. 3c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$

$$\begin{aligned}
\frac{1}{n(n+1)(n+2)} &= \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2} \\
1 &= A(n+1)(n+2) + Bn(n+2) + Cn(n+1) \\
1 &= An^2 + 3An + 2A + Bn^2 + 2Bn + Cn^2 + Cn \\
1 &= An^2 + Bn^2 + Cn^2 + 3An + 2Bn + Cn + 2A
\end{aligned}$$

$$\begin{cases} An^2 + Bn^2 + Cn^2 = 0 \\ 3An + 2Bn + Cn = 0 \\ 2A = 1 \end{cases} = \begin{cases} A = \frac{1}{2} \\ B = -1 \\ C = \frac{1}{2} \end{cases}$$

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} &= \sum_{n=1}^{\infty} \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2n+4} \\
&= \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{3} + \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{1}{4} + \frac{1}{10}\right) \\
&\quad + \cdots + \left(\frac{1}{2n-2} + \frac{1}{2n+2} - \frac{1}{n}\right) + \left(\frac{1}{2n} + \frac{1}{2n+4} - \frac{1}{n+1}\right) \\
&= \lim_{n\to\infty} \frac{1}{4} + \frac{1}{2(n+1)(n+2)} \\
&= \frac{1}{4} + 0 \\
&= \frac{1}{4}
\end{aligned}
$$

(i) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots$

Notice that this is equal to the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$. So,

$$
\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \sum_{n=1}^{\infty} \frac{A}{2n-1} + \frac{B}{2n+1}
$$

$$\Downarrow$$

$$1 = 2An + 2Bn + A - B$$

$$
\begin{cases} 2An + 2Bn = 0 \\ A - B = 1 \end{cases} \implies \begin{cases} A = \frac{1}{2} \\ B = \frac{-1}{2} \end{cases}
$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{1}{4n-2} - \frac{1}{4n+2} \\
&= \left(\frac{1}{2} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{6}} - \cancel{\frac{1}{10}}\right) + \\
&\cdots + \left(\cancel{\frac{1}{4n-5}} - \cancel{\frac{1}{4n-2}}\right) + \left(\cancel{\frac{1}{4n-2}} - \frac{1}{4n+2}\right) \\
&= \lim_{n\to\infty} \frac{1}{2} - \frac{1}{4n+2} \\
&= \frac{1}{2} - 0 \\
&= \frac{1}{2}
\end{aligned}
$$

(j) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+2+3+\cdots+n}$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{1+2+3+\cdots+n} &= \sum_{n=1}^{\infty} \frac{1}{\sum_{i=1}^{n} i} \\ &= \sum_{n=1}^{\infty} \frac{2}{n(n+1)} \\ &\Downarrow \\ \frac{2}{n(n+1)} &= \frac{A}{n} + \frac{B}{n+1} \\ 2 &= An + A + Bn\end{aligned}$$

$$\begin{cases} An + Bn = 0 \\ Bn = 2 \end{cases} = \begin{cases} A = 2 \\ B = -2 \end{cases}$$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{2}{n(n+1)} &= \sum_{n=1}^{\infty} \frac{2}{n} - \frac{2}{n+1} \\ &= \left(\frac{2}{1} - \cancel{\frac{2}{2}}\right) + \left(\cancel{\frac{2}{2}} - \cancel{\frac{2}{3}}\right) + \left(\cancel{\frac{2}{3}} - \cancel{\frac{2}{4}}\right) + \\ &\cdots + \left(\cancel{\frac{2}{n-1}} - \cancel{\frac{2}{n}}\right) + \left(\cancel{\frac{2}{n}} - \frac{2}{n+1}\right) \\ &= \lim_{n\to\infty} 2 - \frac{2}{n+1} \\ &= 2 - 0 \\ &= 2\end{aligned}$$

2. Prove that each of the following series diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n+1}$

Proof. Recall *Theorem 3.7.1 – The n^{th}-Term Test*:

Theorem (The nth Term Test). If the series $\sum x_n$ converges, then $\lim(x_n) = 0$.

Let a_n be the sequence whose terms are obtained by $a_n := \frac{n}{2n+1}$, for $n \in \mathbb{N}$. Then we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2} \text{ by L'Hospital's Rule} = \frac{1}{2} \neq 0$$

Thus since $\displaystyle\lim_{n\to\infty} a_n \neq 0$, we know that by *Theorem 3.7.1*, $\displaystyle\sum_{n=1}^{\infty}$ is divergent. ■

(b) $\displaystyle\sum_{n=1}^{\infty} \cos \frac{1}{n^2}$

Proof. Let a_n be the sequence whose terms are obtained by $a_n := \cos\left(\frac{1}{n^2}\right)$, for $n \in \mathbb{N}$. Then we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos\left(\frac{1}{n^2}\right) = \cos(0) = 1$$

By *The nth Term Test*, since $\lim_{n\to\infty} a_n \neq 0$, the series $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{1}{n^2}\right)$ is divergent. ■

(c) $\displaystyle\sum_{n=1}^{\infty} n \sin \frac{1}{n}$

Proof. Let a_n be the sequence whose terms are obtained by $a_n := n\sin\left(\frac{1}{n}\right)$, for $n \in \mathbb{N}$. Then we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{-\cos\left(\frac{1}{n}\right)}{n^2}}{-\frac{1}{n^2}} = \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos(0) = 1$$

By using *L'Hospital's Rule*. Thus by the *nth Term Test*, since $\lim_{n\to\infty} a_n \neq 0$, the sum $\displaystyle\sum_{n=1}^{\infty} n\sin\left(\frac{1}{n}\right)$ is divergent. ■

(d) $\displaystyle\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$

Proof. Let a_n be the sequence whose terms are obtained by $a_n := \left(1 - \frac{1}{n}\right)^n$, for

$n \in \mathbb{N}$. Then, we have

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n \\
&= \lim_{n\to\infty} e^{\ln\left(1-\frac{1}{n}\right)^n} \\
&= \lim_{n\to\infty} \exp\left\{\ln\left(1 - \frac{1}{n}\right)^n\right\} \\
&= \lim_{n\to\infty} \exp\left\{n\ln\left(1 - \frac{1}{n}\right)\right\} \\
&= \lim_{n\to\infty} \exp\left\{\frac{\ln\left(1 - \frac{1}{n}\right)}{\frac{1}{n}}\right\} \\
&= \lim_{n\to\infty} \exp\left\{\frac{\frac{1}{1-\frac{1}{n}} \cdot \frac{1}{n^2}}{-\frac{1}{n^2}}\right\} \\
&= \lim_{n\to\infty} \exp\left\{\frac{\frac{1}{n^2-n}}{-\frac{1}{n^2}}\right\} \\
&= \lim_{n\to\infty} \exp\left\{-\frac{n^2}{n^2-n}\right\} \\
&= \lim_{n\to\infty} \exp\left\{-\frac{n}{n-1}\right\} \\
&= \lim_{n\to\infty} \exp\left\{-\frac{1}{1-\frac{1}{n}}\right\} \\
&= \exp\left\{-\frac{1}{1-0}\right\} \\
&= \exp(-1) \\
&= e^{-1} \\
&= \frac{1}{e}
\end{aligned}$$

By using *L'Hospital's Rule*. Thus by the *nth Term Test*, since $\lim_{n\to\infty} a_n \neq 0$, we have that the sum $\sum_{n=1}^{\infty}\left(1 - \frac{1}{n}\right)^n$ diverges. ■

3. **(a)** Give an example of two series $\sum a_k$ and $\sum b_k$ that differ in the first five terms, yet converge to the same value.

Consider $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$. This series converges to 2. Also notice that $1 + 2 + 3 + 4 - 10\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = 2$. Thus the first five terms are different but converge to the same value.

(b) Give an example of two series $\sum a_k$ and $\sum b_k$ that differ in infinitely many terms, yet converge to the same value.

Consider the sums

$$\sum_{n=0}^{\infty} \frac{15}{32}\left(\frac{1}{16}\right)^n \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{1}{n(n+1)}$$

Let a_n and b_n be the sequences whose terms are obtained by $a_n := \frac{15}{32}\left(\frac{1}{16}\right)^n$, for $n = 0, 1, 2, 3, \ldots$, and let $b_n := \frac{1}{n(n+1)}$, for $n \in \mathbb{N}$. Then we have

$$a_n := \left(\frac{15}{32}, \frac{15}{512}, \frac{15}{8192}, \frac{15}{131072}, \frac{15}{2097152}, \ldots\right)$$

and

$$b_n := \left(\frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \frac{1}{30}, \frac{1}{42}, \ldots\right)$$

It is clear that since the numerator of each term of a_n is always 15, and that the numerator of b_n is always 1, these two sequences are different at every term and thus differ in infinitely many terms, and thus the terms of the sums $\sum a_n$ and $\sum b_n$ also differ in infinitely many terms. Thus the first five terms of $\sum a_n$ are

$$\frac{15}{32}, \frac{255}{512}, \frac{4095}{8192}, \frac{65535}{131072}, \frac{1048575}{2097152}, \ldots$$

and the first five terms of $\sum b_n$ are

$$\frac{1}{6}, \frac{1}{4}, \frac{3}{10}, \frac{1}{3}, \frac{5}{14}, \ldots$$

However, notice that $\sum a_n$ and $\sum b_n$ converge to the same value:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{15}{32}\left(\frac{1}{16}\right)^n &= \frac{\frac{15}{32}}{1 - \frac{1}{16}} \\
&= \frac{\frac{15}{32}}{\frac{15}{16}} \\
&= \frac{15 \cdot 16}{15 \cdot 32} \\
&= \frac{240}{480} \\
&= \frac{1}{2}
\end{aligned}$$

and

$$\sum_{n=2}^{\infty} \frac{1}{n(n+1)} = \sum_{n=2}^{\infty} \frac{A}{n} + \frac{B}{n+1}$$

$$1 = An + Bn + A$$

$$\begin{cases} An + Bn = 0 \\ A = 1 \end{cases} \implies \begin{cases} A = 1 \\ B = -1 \end{cases}$$

$$\Downarrow$$

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{1}{n(n+1)} &= \sum_{n=2}^{\infty} \frac{1}{n} - \frac{1}{n+1} \\ &= \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \\ &\cdots + \left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right) \\ &= \lim_{n\to\infty} \frac{1}{2} - \frac{1}{n+1} \\ &= \frac{1}{2} - 0 \\ &= \frac{1}{2} \end{aligned}$$

Thus we have that

$$\sum_{n=0}^{\infty} \frac{15}{32} \left(\frac{1}{16}\right)^n = \sum_{n=2}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2}$$

(c) Give an example of two series $\sum a_k$ and $\sum b_k$ that converge to real numbers A and B, respectively, but the series $\sum a_k b_k$ converges to a value different from AB.

Consider the series $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ and $\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$. Then we have

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = \frac{2}{1} = 2$$

and

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$$

Note that $2 \cdot \frac{3}{2} = \frac{6}{2} = 3$. But the series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \left(\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2} \cdot \frac{1}{3}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{6}\right)^n = \frac{1}{1-\frac{1}{6}} = \frac{6}{5}$$

Thus we have that the product of the sums, 3, is not equal to the sum of the products, $\frac{6}{5}$.

(d) Give an example of a series that diverges and whose sequence of partial sums is bounded.

Consider an alternating series, $\sum_{n=1}^{\infty}(-1)^n$. Then, note that $S_1 := -1,\ S_2 := -1+1 = 0,\ S_3 := -1+1-1 = -1, S_4 := -1+1-1+1 = 0, \ldots$. Then we have that the sequence of partial sums is bounded below by -1 and is bounded above by 1. However, since this is an alternating series, we know that by the *Geometric Series Test*, since $|r| = |-1| = 1 \not< 1$, this series is divergent.

4. Prove or justify, if true. Provide a counterexample, if false.

(a) If a_n is strictly decreasing and $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges.

This is a false statement. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n}$. Then, we have that $\lim_{n\to\infty} \frac{1}{n} = 0$, however since this is a harmonic series, we know that it is divergent, thus $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ is divergent.

(b) If $a_n \neq b_n$ for all $n \in \mathbb{N}$ and if $\sum(a_n + b_n)$ converges, then either $\sum a_n$ converges or $\sum b_n$ converges.

This is a false statement. Consider the series $\sum_{n=1}^{\infty}(-1)^n$ and $\sum_{n=1}^{\infty}(-1)^{n+1}$. Then we have that $\sum_{n=1}^{\infty}(-1)^n = -1+1-1+1-1+\ldots$, which is divergent by the *Geometric Series Test*, and $\sum_{n=1}^{\infty}(-1)^{n+1} = 1-1+1-1+1-1+\ldots$, which is also divergent by the *Geometric Series Test*. However, $\sum_{n=1}^{\infty}(-1)^n + (-1)^{n+1} = 0+0+0+0+\cdots = 0$ and thus converges. However, $a_n \neq b_n$ for all $n \in \mathbb{N}$ since $a_n := -1, 1, -1, 1, -1, 1, \ldots$ and $b_n := 1, -1, 1, -1, 1, -1, \ldots$. Thus for all $n \in \mathbb{N}$, either $a_n = -1 \neq 1 = b_n$, or $a_n = 1 \neq -1 = b_n$. Thus we have that $\sum(a_n + b_n)$ converges but neither $\sum a_n$ nor $\sum b_n$ converge.

(c) Suppose $\sum(a_n + b_n)$ converges. Then $\sum a_n$ converges if and only if $\sum b_n$ converges.

This is a true statement since if $\sum(a_n + b_n)$ converges, then both $a_n + b_n$ converges, as was covered in our notes.

(d) If $\lim_{n\to\infty} a_n = A$, then $\sum_{n=1}^{\infty}(a_n - a_{n+2}) = a_1 + a_2 - 2A$.

Proof. Notice that we can rewrite the sum $\sum_{n=1}^{\infty}(a_n - a_{n+2})$ as $\sum_{n=1}^{\infty}((a_n - a_{n+1}) + (a_{n+1} - a_{n+2}))$. Now, we have that the nth partial sum yields a telescoping series:

$$S_n := [(a_1 - \cancel{a_2}) + (a_2 - \cancel{a_3})] + [(\cancel{a_2} - \cancel{a_3}) + (\cancel{a_3} - \cancel{a_4})] + \cdots + [(\cancel{a_n} - a_{n+1}) + (\cancel{a_{n+1}} - a_{n+2})]$$

So $S_n = a_1 + a_2 - a_{n+1} - a_{n+2}$, which yields $\lim_{n\to\infty} = a_1 + a_2 - A - A = a_1 + a_2 - 2A$. ■

(e) $\sum a_n$ converges if and only if $\lim_{n\to\infty} a_n = 0$.

This is a false statement. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n}$. This is the harmonic series, which we know diverges. However, $\lim_{n\to\infty} \frac{1}{n} = 0$. Thus the limit is equal to 0, but the series does not converge.

(f) Changing the first few terms in a series may affect the value of the sum of the series.

Proof. Suppose $x_n \to x$. Then for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that if $n \geq N$, then $|x_n - x| < \varepsilon$. Now, suppose x'_n is a sequence such that for $n \geq M$, then $x'_n = x_n$.

Let $\varepsilon > 0$ be given. Then there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - x| < \varepsilon$. Let $N' = \max(N, M)$. Then if $n \geq N'$, we have that $|x'_n - x| < \varepsilon$. Hence $x'_n \to x$.

Consider a convergent series $\sum x_n$. If we let $s_n = x_1 + x_2 + \cdots + x_n$, then we have that $s_n \to s$.

Consider the series $\sum x'_n$, where for $n \geq M$, then $x'_n = x_n$. Let $s'_n = x'_1 + x'_2 + \cdots + x'_n$. Note that for $n \geq M$, we have $s'_n - s'_{M-1} = x'_M + \cdots + x'_n = x_M + \cdots + x_n$, and thus $s'_n - s'_{M-1} = s_n - s_{M-1} \to s - s_{M-1}$. Hence $s'_n \to (s - s_{M-1} + s'_{M-1})$. ■

(g) Changing the first few terms in a series may affect whether or not the series converges.

This is a false statement. Consider the telescoping series used previously, given by $\sum_{n=2}^{\infty} \frac{1}{n(n+1)}$. We know this series converges to $\frac{1}{2}$. Consider changing the first few terms as follows:

$$\left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \cdots \to \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \ldots$$

Now we have that this series converges to 1, not $\frac{1}{2}$. However, despite changing the first few terms of the series, we did not change whether or not it converges. This is because the first few terms of a series can only finitely affect the sum. Thus, if a series converges, a finite change to the terms will still create a finite sum. Likewise, if the series diverges, a finite change will not allow the series to converge to a finite sum.

(h) If $\sum a_n$ converges and $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$, then $\sum b_n$ converges.

This is a false statement. Consider the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, which we note is the harmonic series. Then we have that $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$ since it is a geometric series. Thus $\sum a_n$ converges. Also, notice that

$$\lim_{n\to\infty} \left(\frac{\left(\frac{1}{2}\right)^n}{\frac{1}{n}} \right) = 0$$

However, since $\displaystyle\sum b_n = \sum_{n=1}^{\infty} \frac{1}{n}$, which is the harmonic series, we know that it is divergent, and thus if $\sum a_n$ converges and $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$, $\sum b_n$ can still be divergent.