Real Analysis II Homework 4

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1. Evaluate

(a) $\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} \sqrt{t+1}\cos t \; dt$

$$\begin{aligned}
\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} \sqrt{t+1}\cos t \; dt &= \lim_{h\to 0} \frac{\int_0^{x+h} \sqrt{t+1}\cos t \; dt - \int_0^{x} \sqrt{t+1}\cos t \; dt}{h} \\
&= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h}, \text{ where } F(x) := \int_0^x \sqrt{t+1}\cos t \; dt \\
F'(x) &= \frac{d}{dx}\int_0^x \sqrt{t+1}\cos t \; dt \\
&\text{let } h(x) = x,\ g(x) = 0, \text{ and } f(x) := \sqrt{x+1}\cos x \text{ then} \\
F'(x) &= f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x) \\
&= \sqrt{x+1}\cos x \cdot 1 - \sqrt{0+1}\cos 0 \cdot 0 \\
&= \sqrt{x+1}\cos x
\end{aligned}$$

Thus $F'(x) = \sqrt{x+1}\cos x$.

(b) $\lim_{x\to a} \frac{x}{x-a}\int_a^x t^2 \; dt$

$$\begin{aligned}
\lim_{x\to a} \frac{x}{x-a}\int_a^x t^2 \; dt &= \lim_{x\to a} \frac{x\cdot \int_a^x t^2 \; dt}{x-a} \\
&= \lim_{x\to a} \frac{x(x^2\cdot 1 - a^2 \cdot 0)}{1-0} \\
&= a^3
\end{aligned}$$

(c) $\lim_{x\to 0} \frac{1}{x}\int_0^x \sqrt{9+t^2}\ dt$

$$\begin{aligned}
\lim_{x\to 0} \frac{1}{x}\int_0^x \sqrt{9+t^2}\ dt &= \lim_{x\to 0} \frac{\int_0^x \sqrt{9+t^2}\ dt}{x} \\
&= \lim_{x\to 0} \frac{\sqrt{9+x^2}\cdot 1 - \sqrt{9+0^2}\cdot 0}{1}, \quad \text{by L'Hospital's Rule and Leibniz's Rule} \\
&= \lim_{x\to 0} \sqrt{9+x^2} \\
&= \sqrt{9} \\
&= 3
\end{aligned}$$

2. **(a)** Show that $(x^2 \sin x)/2$ is not an antiderivative of $x\cos x$.

Proof. We want to show that $\frac{x^2 \sin x}{2} \neq \int x\cos x$ So, we note that we can use *Theorem 7.3.17 Integration by Parts.* So, let $u = x$ and $dv = \cos x\ dx$. Then

$$du = u'dx = 1 \cdot dx = dx$$

and

$$v = \int \cos x\ dx = \sin x + C$$

for some arbitrary constant C. Then,

$$\begin{aligned}
\int x\cos x\ dx &= uv - \int v\ du \\
&= x\sin x - \int \sin x\ dx \\
&= x\sin x - (-\cos x + C) \\
&= x\sin x + \cos x + C
\end{aligned}$$

for some arbitrary constant C. And thus we have that the antiderivative of $x\cos x = x\sin x + \cos x$, which we note is *not* equal to $\frac{x^2\sin x}{2}$. That is,

$$\int x\cos x = x\sin x + \cos x \neq \frac{x^2 \sin x}{2}$$

$\blacksquare$

(b) If $x^2 \cos x = \int_0^x f(t)\ dt$, find $f(x)$.

Since $\int_0^x f(t)\ dt = x^2\cos x$, we know that $x^2\cos x = F(x)$. Thus, in order to find $f(x)$, we must find $F'(x) = f(x)$. So,

$$\frac{d}{dx}x^2\cos x = 2x\cos(x) - x^2\sin(x)$$

And thus we have that $f(x) = 2x\cos(x) - x^2\sin(x)$.

(c) Let $F(x) = \int_0^x xe^{t^2}\ dt$ for $x \in [0,1]$. find $F''(x)$ for $x \in [0,1]$. (Note: $F'(x) \neq xe^{x^2}$)

$$\begin{aligned} F(x) &= x\int_0^x e^{t^2}\ dt \\ F'(x) &= x \cdot e^{x^2} + \int_0^x e^{t^2}\ dt \\ F''(x) &= x \cdot e^{x^2} \cdot 2x + e^{x^2} + e^{x^2} \\ &= 2(x^2+1)e^{x^2} \end{aligned}$$

So $F''(x) = 2(x^2+1)e^{x^2}$.

3. Suppose f is nonnegative and continuous on $[1,2]$ and that $\int_1^2 x^k f(x)\ dx = 5 + k^2$ for $k = 0, 1, 2$.
Prove each of the following:

(a) $\int_1^4 f(\sqrt{x})\ dx \leq 20$.

Let $u = \sqrt{x}$, then $du = u'dx = \frac{1}{2\sqrt{x}}\ dx$ and thus $dx = 2\sqrt{x}\ du$. So,

$\int_1^4 f(\sqrt{x})\ dx = \int_1^2 f(u)\, 2\, \sqrt{x}\, du = 2\int_1^2 \sqrt{x} f(u)\ du = 2\int_1^2 uf(u)\ du = 2(5{+}1) = 12$

Thus $12 \leq 20$.

(b) $\int_{1/\sqrt{2}}^1 f(1/x^2)\ dx \leq 5/2$.

Substitute $u = \frac{1}{x^2}$ and $du = \frac{-2}{x^3}\ dx$. $\frac{1}{x^3}\ dx = \frac{-1}{2}\ du$.
We know $\frac{1}{\sqrt{2}} \leq x \leq 1$ and $\frac{1}{2\sqrt{2}} \leq x^3 \leq 1$. So $2\sqrt{2} \geq \frac{1}{x^3} \geq 1$. Thus $1 \leq \frac{1}{x^3}$ and $\int_{\frac{1}{\sqrt{2}}}^1 1f\left(\frac{1}{x^2}\right) \leq \int_{\frac{1}{\sqrt{2}}}^1 \frac{1}{x^3} f\left(\frac{1}{x^2}\right)\ dx$. So

$$\int_{\frac{1}{\sqrt{2}}}^1 \frac{1}{x^3} f\left(\frac{1}{x^2}\right)\ dx = \int_2^1 \frac{-1}{2} f(u)\ du = \frac{1}{2}\int_1^2 u^0 f(u)\ du = \frac{1}{2}\cdot 5 = \frac{5}{2}$$

Thus $\int_{\frac{1}{\sqrt{2}}}^1 \frac{1}{x^3} f\left(\frac{1}{x^2}\right)\ dx \leq \frac{5}{2}$.

(c) $\int_0^1 x^2 f(x+1)\ dx = 2$.

Let $u = x+1$. Then $du = u'dx = 1dx = dx$. Then,

$$\int_0^1 x^2 f(x+1)\ dx = \int_1^2 x^2 f(u)\ du$$

$$\begin{aligned}
\int_0^1 x^2 f(x+1)\ dx &= \int_1^2 x^2 f(u)\ du \\
&= \int_1^2 (u-1)^2 f(u)\ du \\
&= \int_1^2 (u^2 - 2u + 1) f(u)\ du \\
&= \int_1^2 u^2 f(u)\ du - 2u f(u) + f(u)\ du \\
&= \int_1^2 u^2 f(u)\ du - 2\int_1^2 u f(u)\ du + \int_1^2 f(u)\ du \\
&= (5+2^2) - 2(5+1^2) + (5+0^2) \\
&= 9 - 12 + 5 \\
&= 2
\end{aligned}$$

$$\therefore \int_0^1 x^2 f(x+1)\ dx = 2.$$

4. Suppose that $f \in \mathcal{R}[1/2, 2]$ and that $\displaystyle\int_{1/2}^1 x^k f(x)\ dx = \int_1^2 x^k f(x)\ dx + 2k^2 = 3 + k^2$ for $k = 0, 1, 2$. Compute the exact values of the following integrals:

(a) $\displaystyle\int_0^1 x^3 f(x^2+1)\ dx$

Let $u = x^2 + 1$. Then $du = u'dx = 2x\ dx$. Thus $dx = \frac{du}{2x}$. Note that $x^2 = u - 1$. So, we have

$$
\begin{aligned}
\int_0^1 x^3 f(x^2+1)\ dx &= \int_0^1 x^3 f(u)\ \frac{du}{2x} \\
&= \int_1^2 x^2 f(u)\ \frac{du}{2} \\
&= \frac{1}{2}\ \int_1^2 (u-1) f(u)\ du \\
&= \frac{1}{2}\ \int_1^2 u f(u) - f(u)\ du \\
&= \frac{1}{2}\ \int_1^2 u f(u)\ du - \frac{1}{2}\ \int_1^2 f(u)\ du \\
&= \frac{1}{2}\ \int_1^2 u f(u)\ du - \frac{1}{2} \cdot (3+0) \\
&= \frac{1}{2}\left[\int_1^2 u f(u)\ du + 3\right] \\
&= \frac{1}{2}\left[\int_1^2 u f(u)\ du + 2 - 5\right] \\
&= \frac{1}{2}[3+1-5] \\
&= -\frac{1}{2}
\end{aligned}
$$

$\therefore\ \int_0^1 x^3 f(x^2+1)\ dx = -\frac{1}{2}.$

(b) $\int_0^{\sqrt{3}/2} \frac{x^3}{\sqrt{1-x^2}} f(\sqrt{1-x^2})\ dx$

Let $u = \sqrt{1-x^2}$. Then $du = u'dx = \frac{-2x}{2\sqrt{1-x^2}}\ dx$. Thus $dx = \frac{-2\sqrt{1-x^2}}{2x}\ du$. So,

$$\begin{aligned}
\int_0^{\frac{\sqrt{3}}{2}} \frac{x^3}{\sqrt{1-x^2}} f(\sqrt{1-x^2})\ dx &= \int_1^{\frac{1}{2}} \frac{x^3}{\cancel{u}} f(u) \cdot \frac{-2\cancel{u}}{2x}\ du \\
&= \int_1^{\frac{1}{2}} \frac{-x^3 f(u)}{x}\ du \\
&= \int_1^{\frac{1}{2}} -x^2 f(u)\ du \\
&= -\int_{\frac{1}{2}}^1 (u^2-1) f(u)\ du \\
&= \int_{\frac{1}{2}}^1 (1-u^2) f(u)\ du \\
&= \int_{\frac{1}{2}}^1 f(u)\ du - \int_{\frac{1}{2}}^1 u^2 f(u)\ du \\
&= (3+0^2) - (3+2^2) \\
&= 3-7 \\
&= -4
\end{aligned}$$

$$\therefore \int_0^{\frac{\sqrt{3}}{2}} \frac{x^3}{\sqrt{1-x^2}} f(\sqrt{1-x^2})\ dx = -4.$$

5. Suppose that f, g are differentiable on $[0, e]$ and that $f', g' \in \mathcal{R}[0, e]$.

(a) If $\displaystyle\int_1^e \frac{f(x)}{x}\ dx < f(e)$, prove that $\displaystyle\int_1^e f'(x) \ln x\ dx > 0$.

$$\int_1^e f'(x) \cdot \ln(x)\ dx = f(x)\ln(x)|_1^e - \int_1^e f(x) \cdot \frac{1}{x}\ dx$$

where $u = \ln(x)$, $dv = f'(x)\ dx$, $du = \frac{1}{x}\ dx$, $v = f(x)$. So

$$\int_1^e f'(x) \cdot \ln(x)\ dx = f(x)\ln(x)\Big|_1^e - \int_1^e f(x) \cdot \frac{1}{x}\ dx = f(e) - \int_1^e \frac{1}{x}\ dx > 0$$

Since $\int_1^e \frac{f(x)}{x}\ dx < f(e)$.

(b) If $f(0) = f(1) = 0$, prove that $\displaystyle\int_0^1 e^x[f(x) + f'(x)]\ dx = 0$.

Proof.

$$\int_0^1 e^x\left[f(x) + f'(x)\right]\ dx = \int_0^1 e^x f(x)\ dx + \int_0^1 e^x f'(x)\ dx$$

Let us use *Integration by Parts* on the second integral containing $f'(x)$. Let $u = e^x$. Then $du = e^x\ dx$, $dv = f'(x)\ dx$, and $v = f(x)$. Then we have the following:

$$\begin{aligned}
\int_0^1 e^x f(x)\ dx + \int_0^1 f'(x)\ dx &= \cancel{\int_0^1 e^x f(x)\ dx} + e^x f(x)\Big|_0^1 - \cancel{\int_0^1 f(x) e^x\ dx} \\
&= e^1 f(1) - e^0 f(0) \\
&= e \cdot 0 - 1 \cdot 0 \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

$\therefore \displaystyle\int_0^1 e^x \left[f(x) + f'(x)\right]\ dx = 0.$ ■

6. **(a)** Let $f : [0, b] \to \mathbb{R}$, $b > 0$ be continuous and $f(x) \neq 0$ for all $x \in (0, b)$. Further, suppose $[f(x)]^2 = 2\displaystyle\int_0^x f(t)\ dt$ for all $x \in [0, b]$. Prove that $f(x) = x$ for all $x \in [0, b]$.

We have

$$2f(x)f'(x) = 2f(x)$$

which implies $f(x)[f'(x) - 1] = 0$. Since $f(x) \neq 0$, then $f'(x) - 1 = 0$. So $f'(x) = 1$ and $f(x) = x + C$ for some arbitrary constant C. But $f(0) = 0$ since $[f(0)]^2 = 0 \implies f(0) = 0$. So $f(x) = x$.

(b) Suppose that f is defined on $[0, 1]$ with $f(0) = 0$ and $0 < f'(x) \leq 1$. Prove that $\left[\displaystyle\int_0^1 f(x)\ dx\right]^2 \geq \displaystyle\int_0^1 [f(x)]^3\ dx$.

Proof. Let $x \in [0, 1]$. Then $F(x) := \left[\int_0^x f\right]^2 - \int_0^3 f^3$. So $f(0) = 0$. Thus $F'(x) = 2\left[\int_0^x f\right] \cdot f(x) - f^3(x) = f(x)\left[2\int_0^x f - f^2\right]$ since $f(0) = 0$ and $0 \leq f'(x) \leq 1$ which implies that f is strictly increasing. So $f(x) \geq 0$. ■