Real Analysis II Homework 3

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1. 16. If f is continuous on $[a, b], a < b$, show that there exists $c \in [a, b]$ such that we have $\displaystyle\int_a^b f = f(c)(b - a)$. This result is sometimes called the *Mean Value Theorem for Integrals.*

Proof. Let $m := \inf\{f(x) : x \in [a, b]\}$ and $M := \sup\{f(x) : x \in [a, b]\}$. Then we know from *Theorem 7.1.5 (c)* that

$$m(b - a) \leq \int_a^b f \leq M(b - a)$$

Then, dividing the inequality by $(b - a) > 0$, we have

$$m \leq \frac{\int_a^b f}{b - a} \leq M$$

By *Bolzano's Theorem*, we can conclude that there exists $c \in [a, b]$ s.t.

$$f(c) := \frac{\int_a^b f}{b - a}$$

which can be equivalently written as

$$\int_a^b f = f(c)(b - a)$$

■

19. Suppose that $a > 0$ and that $f \in \mathcal{R}[-a, a]$.

(a) If f is *even* (that is, if $f(-x) = f(x)$ for all $x \in [0, a]$), show that $\displaystyle\int_{-a}^a f = 2\int_0^a f$.

Proof. Since f is even, we have

$$\begin{aligned}\int_{-a}^b f &= \int_{-a}^0 f(x)\ dx + \int_0^a f(x)\ dx \\ &= -\int_a^0 f(-y)\ dy + \int_0^a f(x)\ dx\end{aligned}$$

where $y = -x$ for the first integral. Thus $x \mapsto -y,\ -a \mapsto a,\ 0 \mapsto 0.$

$$= -\int_a^0 f(y)\ dy + \int_0^a f(x)\ dx \qquad (f \text{ is even so } f(-y) = f(y)$$
$$= \int_0^a f(y)\ dy + \int_0^a f(x)\ dx$$

since flipping the limits of integration changes the sign of the integral

$$= 2\int_0^a f(x)$$
$$= 2\int_0^a f$$

■

(b) If f is *odd* (that is, if $f(-x) = -f(x)$ for all $x \in [0, a]$), show that $\int_{-a}^a f = 0$.
Proof. Since f is odd, we have

$$\int_{-a}^a f = \int_{-a}^0 f(x)\ dx + \int_0^a f(x)\ dx$$
$$= -\int_a^0 f(-y)\ dy + \int_0^a f(x)\ dx$$

where $y = -x$, thus giving us $x \mapsto -y,\ -a \mapsto a,\ 0 \mapsto 0.$

$$= -\int_a^0 (-f(y))\ dy + \int_0^a f(x)\ dx \qquad \text{since } f \text{ is odd, } f(-x) = -f(x)$$
$$= \int_0^a (-f(y))\ dy + \int_0^a f(x)\ dx$$
$$= -\int_0^a f(y)\ dy + \int_0^a f(x)\ dx$$

since flipping the limits of integration changes the sign of the integral,

$$= 0$$

since both integrals cancel each other out. ■

20. If f is continuous on $[-a, a]$, show that $\int_{-a}^a f(x^2)dx = 2\int_0^a f(x^2)dx$.

Proof. Since f is continuous on $[-a, a]$, and x^2 is a continuous function, we know that $f(x^2)$ is a continuous function since the composition of continuous functions is continuous. That is, $f(x^2)$ is a continuous function since it is the composition of the continuous functions$f(x)$, and $x \mapsto x^2$.

Let $g : [a, b] \to \mathbb{R}$ be given by $g(x) := f(x^2)$, then g is also continuous as it is a composition of the continuous functions $x \mapsto f(x)$ and $x \mapsto x^2$. Notice, however, that $g(-x) = f((-x)^2) = f(x^2) = g(x)$. This means that g is an even function.

So by the preceding problem, we know that

$$\int_{-a}^{a} g(x)\ dx = 2\int_{0}^{a} g(x)\ dx$$

Therefore

$$\int_{-a}^{a} f(x^2)\ dx = 2\int_{0}^{a} f(x^2)\ dx$$

■

2. **3.** If $g(x) := x$ for $|x| \geq 1$ and $g(x) := -x$ for $|x| < 1$ and if $G(x) := \frac{1}{2}|x^2 - 1|$, show that $\int_{-2}^{3} g(x) dx = G(3) - G(-2) = \frac{5}{2}$. Also sketch the graphs of g and G.

Proof. Let $g(x) := \begin{cases} x, & |x| \geq 1 \\ -x, & |x| < 1 \end{cases}$, and let $G(x) := \frac{1}{2}\left|x^2 - 1\right|$. We want to show that $\int_{-2}^{3} g(x)\ dx = G(3) - G(-2) = \frac{5}{2}$.

Notice that since G is a composition of continuous functions, namely $|x^2 - 1|$ and x^2, we know that G is also continuous. Also, notice that g has a finite number of discontinuities, namely at $x = -1$ and $x = 1$. Thus, $g \in \mathcal{R}[-2, 3]$.

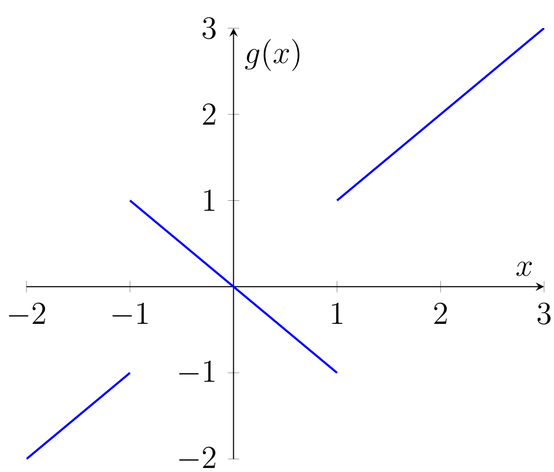
Lastly, note that

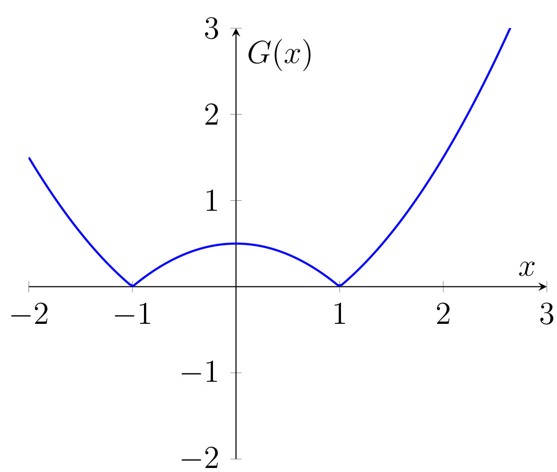
$$G'(x) := \begin{cases} x, & |x| \geq 1 \\ -x, & |x| < 1 \end{cases} = g(x),\ \forall\ x \in [-2, 3] \setminus \{-1, 1\}$$

Thus, by the Fundamental Theorem of Calculus, we have

$$\int_{-2}^{3} g(x)\ dx = G(3) - G(-2) = \frac{1}{2}|9 - 1| - \frac{1}{2}|4 - 1| = \frac{8}{2} - \frac{3}{2} = \frac{5}{2}$$

■





9. Let $f \in \mathcal{R}[a,b]$ and define $F(x) := \displaystyle\int_a^x f$ for $x \in [a,b]$.

(a) Evaluate $G(x) := \displaystyle\int_c^x f$ in terms of F, where $c \in [a,b]$.

$$\begin{aligned} G(x) &= \int_c^x f \\ &= \int_a^c f + \int_c^x f - \int_a^c f \\ &= \int_a^x f - \int_a^c f \\ &= F(x) - F(c) \end{aligned}$$

(b) Evaluate $H(x) := \displaystyle\int_x^b f$ in terms of F.

$$\begin{aligned} H(x) &= \int_x^b f \\ &= \int_a^x f + \int_x^b f - \int_a^x f \\ &= F(b) - F(x) \end{aligned}$$

(c) Evaluate $S(x) := \displaystyle\int_x^{\sin x} f$ in terms of F.

$$\begin{aligned} S(x) &= \int_x^{\sin x} f \\ &= \int_a^x f + \int_x^{\sin x} f - \int_a^x f \\ &= \int_a^{\sin x} f - \int_a^x f \\ &= F(\sin x) - F(x) \end{aligned}$$

11. Find $F'(x)$ when F is defined on $[0,1]$ by:

(a) $F(x) := \displaystyle\int_0^{x^2} (1+t^3)^{-1}\, dt.$

Since x^2 is continuous and differentiable on $[0,1]$, we can use *Leibniz's Rule* to find $F'(x)$, where $f(x) = \frac{1}{1+x^3}$, $h(x) := x^2$, and $g(x) := 0$. So,

$$\begin{aligned} F'(x) &= f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x) \\ &= \frac{1}{1+(x^2)^3} \cdot 2x - \frac{1}{1+(0)^3} \cdot 0 \\ &= \frac{2x}{1+x^6} - 0 \\ &= \frac{2x}{1+x^6} \end{aligned}$$

(b) $F(x) := \displaystyle\int_{x^2}^{x} \sqrt{1+t^2}\ dt.$

Since both x and x^2 are continuous and differentiable on $[0,1]$, we can use *Leibniz's Rule* to find $F'(x)$, where $f(x) := \sqrt{1+x^2}$, $h(x) := x$, and $g(x) := x^2$. So, we must first rewrite $F(x)$ as

$$F(x) := \int_{x^2}^{x} \sqrt{1+x^2}\ dx = \int_0^x \sqrt{1+x^2}\ dx - \int_0^{x^2} \sqrt{1+x^2}\ dx$$

$$\begin{aligned} F'(x) &= f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x) \\ &= \sqrt{1+x^2} \cdot 1 - \sqrt{1+(x^2)^2} \cdot 2x \\ &= \sqrt{1+x^2} - 2x \cdot \sqrt{1+x^4} \end{aligned}$$

12. Let $f : [0,3] \to \mathbb{R}$ be defined by $f(x) := x$ for $0 \le x < 1$, $f(x) := 1$ for $1 \le x < 2$ and $f(x) := x$ for $2 \le x \le 3$. Obtain formulas for $F(x) := \int_0^x f$ and sketch the graphs of f and F. Where is F differentiable? Evaluate $F'(x)$ at all such points.

Let $f(x) := \begin{cases} x, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ x, & 2 \le x \le 3 \end{cases}$

Then we have the following: When $x \in [0,1)$:

$$\begin{aligned} F(x) &= \int_0^x f(t)\ dt \\ &= \int_0^x t\ dt \\ &= \frac{x^2}{2} \end{aligned}$$

When $x \in [1, 2)$:

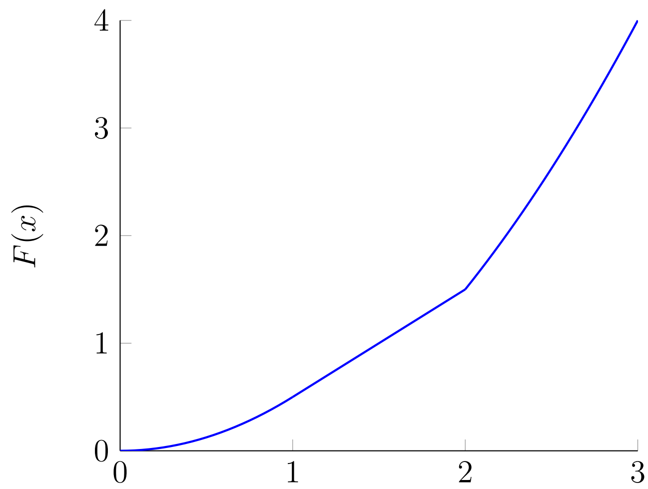
$$\begin{aligned} F(x) &= \int_0^x f(t)\ dt \\ &= \int_0^1 t\ dt + \int_1^x 1\ dt \\ &= \frac{1}{2} + (x - 1) \\ &= x - \frac{1}{2} \end{aligned}$$

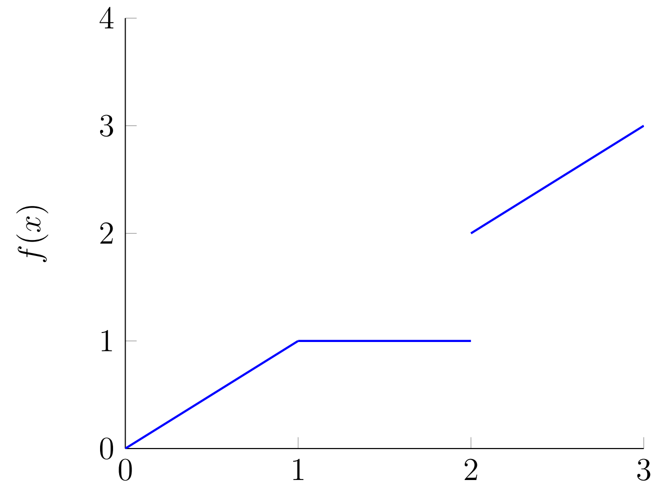
When $x \in [2, 3]$:

$$\begin{aligned} F(x) &= \int_0^x t\ dt \\ &= \int_0^1 t\ dt + \int_1^2 1\ dt + \int_2^3 t\ dt \\ &= \frac{1}{2} + 1 + \left(\frac{x^2}{2} - \frac{2^2}{2}\right) \\ &= \frac{x^2}{2} - \frac{1}{2} \end{aligned}$$

Therefore, we have

$$F(x) = \begin{cases} \frac{x^2}{2}, & 0 \le x < 1 \\ x - \frac{1}{2}, & 1 \le x < 2 \\ \frac{x^2}{2} - \frac{1}{2}, & 2 \le x \le 3 \end{cases}$$





F is definitely differentiable at points $x \in (0,1) \cup (1,2) \cup (2,3)$ since at those points, f is equal to a polynomial. So, we must now check for the differentiability of f at $x = 1$ and $x = 2$.

$$\lim_{x \to 1^-} \frac{F(x) - F(1)}{x - 1} = \lim_{x \to 1^-} \frac{\frac{x^2}{2} - \frac{1}{2}}{x - 1} = \frac{1}{2} \lim_{x \to 1^-} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = \frac{1}{2} \cdot (1+1) = 1$$

$$\lim_{x \to 1^+} \frac{F(x) - F(1)}{x - 1} = \lim_{x \to 1^+} \frac{\left(x - \frac{1}{2}\right) - \frac{3}{2}}{x - 1} = 1$$

Therefore, F is differentiable at $x = 1$ and $F'(x) = 1$. As for when $x = 2$,

$$\lim_{x \to 2^-} \frac{F(x) - F(2)}{x - 1} = \lim_{x \to 2^-} \frac{\left(x - \frac{1}{2}\right) - \frac{3}{2}}{x - 2} = 1$$

$$\lim_{x \to 2^+} \frac{F(x) - F(2)}{x - 2} = \lim_{x \to 2^+} \frac{\left(\frac{x^2}{2} - \frac{1}{2}\right) - \frac{1}{2}}{x - 2} = 2$$

Therefore, F is not differentiable at $x = 2$. Thus,

$$F'(x) := \begin{cases} x, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ x, & 2 < x \le 3 \end{cases}$$

And notice that $F'(x) = f(x)$ for $x \in [0,1] \setminus \{2\}$

13. The function g is defined on $[0,3]$ by $g(x) := -1$ if $0 \le x < 2$ and $g(x) := 1$ if $2 \le x \le 3$. Find the indefinite integral $G(x) = \int_0^x g$ for $0 \le x \le 3$, and sketch the graphs of g and G. Does $G'(x) = g(x)$ for all $x \in [0,3]$?

$$g(x) := \begin{cases} -1, & 0 \le x < 2 \\ 1, & 2 \le x \le 3 \end{cases}$$

Then we have the following: when $x \in [0, 2)$:

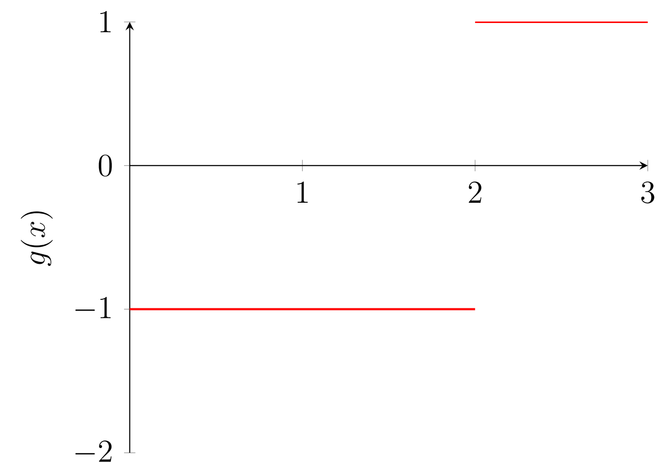
$$\begin{aligned} G(x) &= \int_0^x g(t)\ dt \\ &= \int_0^x -1\ dt \\ &= -x \end{aligned}$$

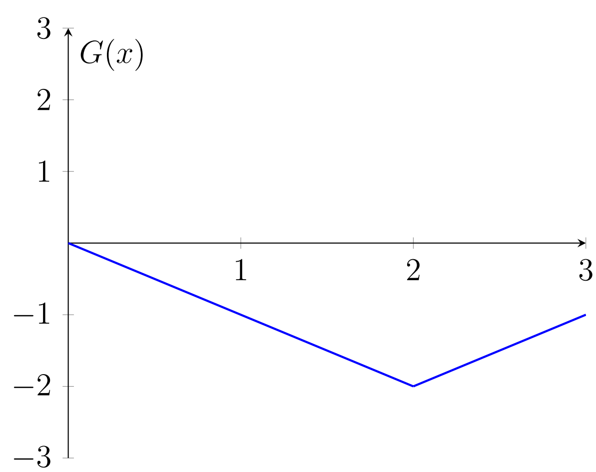
When $x \in [2, 3]$:

$$\begin{aligned} G(x) &= \int_0^x g(t)\ dt \\ &= \int_0^2 -1\ dt + \int_2^x 1\ dt \\ &= -2 + x - 2 \\ &= x - 4 \end{aligned}$$

Therefore,

$$F(x) = \begin{cases} -x, & 0 \le x < 2 \\ x - 4, & 2 \le x \le 3 \end{cases}$$





Now, if it is possible, let $G'(x) = g(x)\ \forall\ x \in [0,3]$. Then

$$\lim_{x\to 2^-} \frac{G(x)-G(2)}{x-2} = \lim_{x\to 2^-} \frac{-x+2}{x-2} = -1$$

$$\lim_{x\to 2^+} \frac{G(x)-G(2)}{x-2} = \lim_{x\to 2^+} \frac{x-2}{x-2} = 1$$

So,

$$\lim_{x\to 2^-} \frac{G(x)-G(2)}{x-2} \neq \lim_{x\to 2^+} \frac{G(x)-G(2)}{x-2}$$

Thus we have that the limit does not exist, and hence G is not differentiable at $x = 2$. Thus $G'(x) \neq g(x)$ for some $x \in [0,3]$.

16. If $f : [0,1] \to \mathbb{R}$ is continuous and $\int_0^x f = \int_x^1 f$ for all $x \in [0,1]$, show that $f(x) = 0$ for all $x \in [0,1]$.

Proof. Let $F(x) = \displaystyle\int_0^x f(t)\,dt$ for $x \in [0,1]$. F is well defined since f is continuous on $[0,1]$, and thus is also integrable on $[0,1]$. F is also differentiable since f is continuous and $F'(x) = f(x)\ \forall\ x \in [0,1]$. So,

$$\begin{aligned}\int_0^x f = \int_x^1 f \Leftrightarrow \int_0^x f &= \int_0^1 f - \int_0^x f \\ \Leftrightarrow 2\int_0^x f &= \int_0^1 f \\ \Leftrightarrow 2F(x) &= F(1)\end{aligned}$$

And differentiating the last relation with respect to x, we get

$$2F'(x) = 0 \Leftrightarrow F'(x) = 0 \Leftrightarrow f(x) = 0,\ \forall\ x \in [0,1]$$

∎

18. Use the Substitution Theorem 7.3.8 to evaluate the following integral:

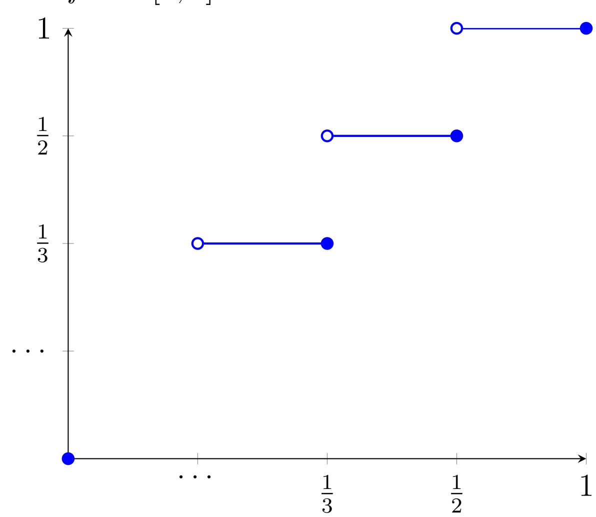
(c) $\int_1^4 \frac{\sqrt{1+\sqrt{t}}}{\sqrt{t}}dt$

Let $\phi(t) = 1+\sqrt{t}$ for $t \in [1,4]$, and let $f(u) = \sqrt{u}$ for $u \in [2,3]$. f is continuous on $[2,3]$ and ϕ has a continuous derivative (namely, $\phi'(t) = \frac{1}{2\sqrt{t}}$) on

$[1,4]$. Thus we have

$$\begin{aligned}
\int_1^4 \frac{\sqrt{1+\sqrt{t}}}{\sqrt{t}}\ dt &= 2\int_1^4 \frac{\sqrt{1+\sqrt{t}}}{\sqrt{t}}\ dt \\
&= 2\int_1^4 \phi'(t)\cdot f(\phi(t))\ dt \\
&= 2\int_{\phi(1)}^{\phi(4)} f(u)\ dt && \text{by the \textit{Substitution Theorem}} \\
&= 2\int_{\phi(1)}^{\phi(4)} \sqrt{u}\ du \\
&= 2\int_2^3 \sqrt{u}\ du \\
&= 2\cdot \frac{2}{3}u^{\frac{3}{2}}\Big|_{u=2}^{u=3} \\
&= \frac{4}{3}\left(3^{\frac{3}{2}} - 2^{\frac{3}{2}}\right)
\end{aligned}$$

3. Let $f:[0,1]\to\mathbb{R}$ given by $f(x) = \begin{cases} \frac{1}{n} & \text{if } x\in\left(\frac{1}{n+1},\frac{1}{n}\right] \\ 0 & \text{if } x = 0\end{cases}$. Sketch the graph of f and show that $f\in\mathcal{R}[0,1]$.



Since f is monotone, by *Theorem 7.2.8*, $f\in\mathcal{R}[0,1]$.

4. **(a)** Give an example of two functions $f,g:[a,b]\to\mathbb{R}$ that are not Riemann integrable, but $fg\in\mathcal{R}[a,b]$.

Consider

$$f(x) := \begin{cases} 1, & x\in\mathbb{Q} \\ 0, & x\in\mathbb{R}\setminus\mathbb{Q}\end{cases} \qquad \text{and} \qquad g(x) := \begin{cases} 0, & x\in\mathbb{Q} \\ 1, & x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}$$

Since these are both Dirichlet and modified Dirichlet functions, we know that they are not Riemann integrable, however,

$$fg := \begin{cases} 0, x \in \mathbb{Q} \\ 0, x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} = 0$$

and thus fg is a constant function, which is Riemann integrable. Thus we have that $f, g \notin \mathcal{R}[a, b]$, but $fg \in \mathcal{R}[a, b]$.

(b) Give an example of two functions $f, g : [a, b] \to \mathbb{R}$ where $f \in \mathcal{R}[a, b]$ and $g \notin \mathcal{R}[a, b]$, but $fg \in \mathcal{R}[a, b]$.

Consider

$$f(x) := 0, \ \forall \ x \in [a, b], \qquad \text{and} \qquad g(x) := \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Then since f is a constant function, $f \in \mathcal{R}[a, b]$, and since g is the Dirichlet function, we know that $g \notin \mathcal{R}[a, b]$. However,

$$fg = \begin{cases} 0, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} = 0$$

And thus fg is a constant function and is thus Riemann integrable. Thus we have that $f \in \mathcal{R}[a, b]$, $g \notin \mathcal{R}[a, b]$, and $fg \in \mathcal{R}[a, b]$.

(c) Let $f : [a, b] \to \mathbb{R}$, $f \in \mathcal{R}[a, b]$. Let $F : [a, b] \to \mathbb{R}$ be given by $F(x) = \displaystyle\int_a^x f(t)dt$. Prove that F is Lipschitz.

Proof. Since $f \in \mathcal{R}[a, b]$, f is bounded; that is, there is some M s.t. $|f(x)| \leq M \ \forall \ x \in [a, b]$. Now, if $y < x$, we have

$$|F(x) - F(y)| = \left| \int_y^x f(t) \ dt \right| \leq \int_y^x |f(t)| \ dt \leq \int_y^x M \ dt = M(x - y) = M|x - y|$$

Similarly, $|F(x) - F(y)| \leq M(y - x) = M|x - y|$ if $y > x$. So, we see that for any $x, y \in [a, b]$, if we let $K = M$, we have

$$|F(x) - F(y)| \leq K|x - y|$$

■

5. Let $f(t) := \begin{cases} t & \text{for } 0 \leq t \leq 2 \\ 3 & \text{for } 2 < t \leq 4 \end{cases}$

(a) Find an explicit expression for $F(x) = \int_0^x f(t)dt$.

When $x \in [0, 2]$:

$$\begin{aligned} F(t) &= \int_0^x f(t)\ dt \\ &= \int_0^x t\ dt \\ &= \frac{x^2}{2} \end{aligned}$$

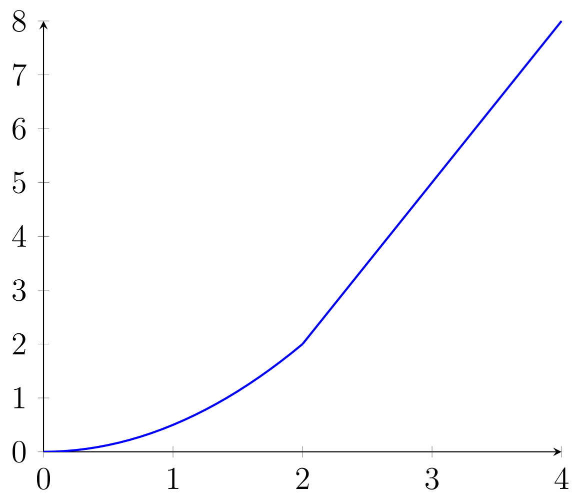
and when $x \in (2, 4]$:

$$\begin{aligned} F(t) &= \int_0^x f(t)\ dt \\ &= \int_0^2 t\ dt + \int_2^x 3\ dt \\ &= 2 + 3x - 6 \\ &= 3x - 4 \end{aligned}$$

Thus,

$$F(x) := \begin{cases} \frac{x^2}{2}, & 0 \le x \le 2 \\ 3x - 4, & 2 < t \le 4 \end{cases}$$

(b) Sketch F and determine where F is differentiable.



Based on the graph, we can tell that the only place in which F is not differentiable is at $x = 2$, which we can see as follows:

$$\lim_{x \to 2^-} \frac{F(x) - F(x)}{x - 2} = \lim_{x \to 2^-} \frac{\frac{x^2}{2} - 2}{x - 2} = 2$$

$$\lim_{x\to 2^+} \frac{F(x)-F(2)}{x-2} = \lim_{x\to 2^+} \frac{3x-4-2}{x-2} = \lim_{x\to 2^+} \frac{3x-2}{x-2} = 3$$

Since $\lim\limits_{x\to 2^-} \dfrac{F(x)-F(2)}{x-2} \neq \lim\limits_{x\to 2^+} \dfrac{F(x)-F(2)}{x-2}$, we have that F is not differentiable when $x = 2$.

(c) Find formula for $F'(x)$ wherever F is differentiable.

Since the only place in which F is not differentiable is when $x = 2$, we need only change one of the inequalities of f. So,

$$F'(x) := \begin{cases} x, & 0 \le x < 2 \\ 3, & 2 < x \le 4 \end{cases}$$

6. Prove or justify, if true or provide a counterexample, if false.

(a) If $f \in \mathcal{R}[a,b]$, then $|f| \in \mathcal{R}[a,b]$ and $\displaystyle\int_a^b |f| \le \left|\int_a^b f\right|$.

This is false since the inequality is flipped from how it appears in *Corollary 7.3.15* to the *Composition Theorem*. Consider $f : [0, 2\pi] \to R$ given by $f(x) := \sin x$. Then we have

$$\left|\int_0^{2\pi} \sin x \; dx\right| = |0| = 0 \le 4 = \int_0^{2\pi} |\sin x| \; dx$$

and thus

$$\left|\int_0^{2\pi} \sin x \; dx\right| \le \int_0^{2\pi} |\sin x| \; dx$$

(b) If $f, g \in \mathcal{R}[a,b]$ and f is continuous, then there exists a $c \in [a,b]$ such that $\displaystyle\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$.

This is a false statement. Consider $f, g : [-1, 1] \to \mathbb{R}$ given by $f(x) := x + 2$ and $g(x) := x$. Then we have the following:

$$\int_{-1}^1 fg = \int_{-1}^1 x^2 + 2x \; dx = \frac{2}{3} \neq 0 \cdot (x+2) = f(c)\int_{-1}^1 x \; dx$$

Thus $\nexists\, c \in [a,b]$ s.t. $\displaystyle\int_{-1}^1 f(x)g(x) = f(c) \cdot \int_{-1}^1 g(x) \; dx$ for f, g as given.

(c) If $f \in C^1(\mathbb{R})$, then $\frac{d}{dx}\displaystyle\int_0^x f(t)dt = \int_0^x \left[\frac{d}{dx} f(t)\right] dt$.

This is a false statement, consider the function $f(x) := \cos x$. Then,

$$\frac{d}{dx}\int_0^x \cos x \; dx = \frac{d}{dx}(\sin x) = \cos x$$

and

$$\int_0^x \left[\frac{d}{dx}\cos x\right] \; dx = \int_0^x -\sin x \; dx = \cos x - 1$$

Hence we have that equality does not hold.

(d) If $f'(x) = \sin x - \cos x$, then $f(x) = \displaystyle\int_0^x (\sin t - \cos t) dt$.

This is a false statement. Note the following:

$$\int_0^x (\sin t - \cos t) \; dt = (-\sin x - \cos x + 1)$$

but

$$f(x) = \int (\sin x - \cos x) \; dx = -\cos x - \sin x$$

Hence

$$-\sin x - \cos x + 1 \neq -\sin x - \cos x$$

(e) If $f, g : [a, b] \to \mathbb{R}$ are such that $f, fg \in \mathcal{R}[a, b]$ and f is strictly monotone on $[a, b]$, then $g \in \mathcal{R}[a, b]$?

This is a false statement. Consider $f, g : [-1, 1] \to \mathbb{R}$ given by $f(x) = x$ and $g(x) = \frac{1}{x}$. Then both $f, fg \in \mathcal{R}[-1, 1]$ since x is integrable and $fg = x \cdot \frac{1}{x} = \frac{x}{x} = 1$ is also integrable. However, since $\lim\limits_{x \to 0^-} g(x) = -\infty$ and $\lim\limits_{x \to 0^+} g(x) = \infty$, we have that $g(x) \notin \mathcal{R}[-1, 1]$.