Real Analysis II Homework 2

Alexander J. Tusa

February 23, 2019

1. Section 7.4

1. Let $f(x) := |x|$ for $-1 \leq x \leq 2$. Calculate $L(f; P)$ and $U(f, P)$ for the following partitions:

(a) $\mathcal{P}_1 := (-1, 0, 1, 2)$

Our terms are:

$$x_0 := -1, \quad x_1 := 0, \quad x_2 := 1, \quad x_3 := 2$$

and our intervals are:

$$I_1 := [-1, 0], \quad I_2 := [0, 1], \quad I_3 := [1, 2]$$

thus $L(f, \mathcal{P}_1)$ is:

$$\begin{aligned}
L(f, \mathcal{P}_1) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} \left(\inf\{f(x) : x \in [x_{i-1}, x_i]\}\right)(x_i - x_{i-1}) \\
&= (\inf\{|x| : x \in [-1, 0]\})(0 - (-1)) \\
&+ (\inf\{|x| : x \in [0, 1]\})(1 - 0) \\
&+ (\inf\{|x| : x \in [1, 2]\})(2 - 1) \\
&= 0 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 \\
&= 0 + 0 + 1 \\
&= 1
\end{aligned}$$

and

$$
\begin{aligned}
U(f, \mathcal{P}_1) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} sup\{f(x) : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} \sup\{|x| : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) \\
&= (\sup\{|x| : x \in [-1, 0]\})(0 - (-1)) \\
&+ (\sup\{|x| : x \in [0, 1]\})(1 - 0) \\
&+ (\sup\{|x| : x \in [1, 2]\})(2 - 1) \\
&= 1 \cdot 1 + 1 \cdot 1 + 2 \cdot 1 \\
&= 1 + 1 + 2 \\
&= 4
\end{aligned}
$$

So, $L(f, \mathcal{P}_1) = 1$ and $U(f, \mathcal{P}_1) = 4$

(b) $\mathcal{P}_2 := (-1, -1/2, 0, 1/2, 1, 3/2, 2)$.

Our terms are:

$$
x_0 := -1, \quad x_1 := -\frac{1}{2}, \quad x_2 := 0, \quad x_3 := \frac{1}{2}, \quad x_4 := 1, \quad x_5 := \frac{3}{2}, \quad x_6 := 2
$$

and our intervals are:

$$
I_1 := \left[-1, -\frac{1}{2}\right], \; I_2 := \left[-\frac{1}{2}, 0\right], \; I_3 := \left[0, \frac{1}{2}\right], \; I_4 := \left[\frac{1}{2}, 1\right], \; I_5 := \left[1, \frac{3}{2}\right], \; I_6 := \left[\frac{3}{2}, 2\right]
$$

So $L(f, \mathcal{P}_2)$ is

$$\begin{aligned}
L(f, \mathcal{P}_2) &= \sum_{i=1}^{n} m_i (x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} \inf\{f(x) : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) \\
&= \inf\left\{|x| : x \in \left[-1, -\frac{1}{2}\right]\right\}\left(-\frac{1}{2} - (-1)\right) \\
&+ \inf\left\{|x| : x \in \left[-\frac{1}{2}, 0\right]\right\}\left(0 - \left(-\frac{1}{2}\right)\right) \\
&+ \inf\left\{|x| : x \in \left[0, \frac{1}{2}\right]\right\}\left(\frac{1}{2} - 0\right) \\
&+ \inf\left\{|x| : x \in \left[\frac{1}{2}, 1\right]\right\}\left(1 - \frac{1}{2}\right) \\
&+ \inf\left\{|x| : x \in \left[1, \frac{3}{2}\right]\right\}\left(\frac{3}{2} - 1\right) \\
&+ \inf\left\{|x| : x \in \left[\frac{3}{2}, 2\right]\right\}\left(2 - \frac{3}{2}\right) \\
&= \frac{1}{2} \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} + \frac{3}{2} \cdot \frac{1}{2} \\
&= \frac{1}{4} + 0 + 0 + \frac{1}{4} + \frac{1}{2} + \frac{3}{4} \\
&= \frac{7}{4}
\end{aligned}$$

and

$$
\begin{aligned}
U(f, \mathcal{P}_2) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} \sup\{f(x) : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) \\
&= \sup\left\{|x| : x \in \left[-1, -\frac{1}{2}\right]\right\}\left(-\frac{1}{2} - (-1)\right) \\
&+ \sup\left\{|x| : x \in \left[-\frac{1}{2}, 0\right]\right\}\left(0 - \left(-\frac{1}{2}\right)\right) \\
&+ \sup\left\{|x| : x \in \left[0, \frac{1}{2}\right]\right\}\left(\frac{1}{2} - 0\right) \\
&+ \sup\left\{|x| : x \in \left[\frac{1}{2}, 1\right]\right\}\left(1 - \frac{1}{2}\right) \\
&+ \sup\left\{|x| : x \in \left[1, \frac{3}{2}\right]\right\}\left(\frac{3}{2} - 1\right) \\
&+ \sup\left\{|x| : x \in \left[\frac{3}{2}, 2\right]\right\}\left(2 - \frac{3}{2}\right) \\
&= 1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} + \frac{3}{2} \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{2} + \frac{3}{4} + 1 \\
&= \frac{13}{4}
\end{aligned}
$$

So $L(f, \mathcal{P}_2) = \frac{7}{4}$ and $U(f, \mathcal{P}_2) = \frac{13}{4}$

2. Prove if $f(x) := c$ for $x \in [a, b]$, then its Darboux integral is equal to $c(b - a)$.

Proof. Let $\mathcal{P} := (x_0, x_1, \ldots, x_n)$ be a partition of $[a, b]$ where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

then $M_i := \sup f(x) = c$ since f is constant, for all $x \in [x_{i-1}, x_i]$, and $m_i := \inf f(x) = c$ again since f is constant, for all $x \in [x_{i-1}, x_i]$.

Then we have that

$$\begin{aligned}
U(f,\mathcal{P}) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} c(x_i - x_{i-1}) \\
&= c\sum_{i=1}^{n} (x_i - x_{i-1}) \\
&= c(x_n - x_0) \\
&= c(b-a), && \text{as both } b \text{ and } a \text{ were defined for } \mathcal{P}
\end{aligned}$$

So $U(f,\mathcal{P}) := c(b-a)$. As for $L(f,\mathcal{P})$:

$$\begin{aligned}
L(f,\mathcal{P}) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} c(x_i - x_{i-1}) \\
&= c\sum_{i=1}^{n} (x_i - x_{i-1}) \\
&= c(x_n - x_0) \\
&= c(b-a), && \text{as both } b \text{ and } a \text{ were defined for } \mathcal{P}
\end{aligned}$$

and thus $L(f,\mathcal{P}) = c(b-a)$.

Now we must find the Darboux integral of $f(x)$. So we have that the upper Darboux integral of $f(x)$ is

$$\begin{aligned}
U(f) &= \inf\{U(f,\mathcal{P}) : \mathcal{P} \in \mathscr{P}[a,b]\} \\
&= \inf\{c(b-a) : \mathcal{P} \in \mathscr{P}[a,b]\} \\
&= c(b-a)
\end{aligned}$$

and the lower Darboux integral is

$$\begin{aligned}
L(f) &= \sup\{L(f,\mathcal{P}) : \mathcal{P} \in \mathscr{P}[a,b]\} \\
&= \sup\{c(b-a) : \mathcal{P} \in \mathscr{P}[a,b]\} \\
&= c(b-a)
\end{aligned}$$

Thus we have that $U(f) = L(f)$, which yields that f is Darboux integrable on $[a,b]$ and the Darboux integral of f is $c(b-a)$. ■

3. Let f and g be bounded functions on $I := [a,b]$. If $f(x) \leq g(x)$ for all $x \in I$, show that $L(f) \leq L(g)$ and $U(f) \leq U(g)$.

Proof. Let f, g be bounded on $I := [a, b]$ such that $f(x) \leq g(x)\ \forall\ x \in I$, and let $\mathcal{P} := (x_0, x_1, \ldots, x_n)$ be a partition of $[a, b]$ where

$$a = x_0 < x_1 < \cdots < x_n = b$$

Let $M_{i_1} := \sup\{f(x) : x \in [x_{i-1}, x_i]\}$, and let $m_{i_1} := \inf\{f(x) : x \in [x_{i-1}, x_i]\}$, and $M_{i_2} := \sup\{g(x) : x \in [x_{i-1}, x_i]\}$ and $m_{i_2} := \inf\{g(x) : x \in [x_{i-1}, x_i]\}$. Then since $f(x) \leq g(x)$, we know that $\sup f(x) \leq \sup g(x)$ and that $\inf f(x) \leq \inf g(x)$. This in turn means that

$$\sup\{f(x) : x \in [x_{i-1}, x_i]\} \leq \sup\{g(x) : x \in [x_{i-1}, x_i]\}$$

and

$$\inf\{f(x) : x \in [x_{i-1}, x_i]\} \leq \inf\{g(x) : x \in [x_{i-1}, x_i]\}$$

which implies that $M_{i_1} \leq M_{i_2}$ and $m_{i_1} \leq m_{i_2}$ for $i = 1, 2, \ldots, n$.

Then we have the following:

$$L(f, \mathcal{P}) = \sum_{i=1}^{n} m_{i_1}(x_i - x_{i-1}) \leq \sum_{i=1}^{n} m_{i_2}(x_i - x_{i-1}) = L(g, \mathcal{P})$$

So $L(f, \mathcal{P}) \leq L(g, \mathcal{P})$ and

$$U(f, \mathcal{P}) = \sum_{i=1}^{n} M_{i_1}(x_i - x_{i-1}) \leq \sum_{i=1}^{n} M_{i_2}(x_i - x_{i-1}) = U(g, \mathcal{P})$$

So $U(f, \mathcal{P}) \leq U(g, \mathcal{P})$.

Now, we have that

$$L(f) = \sup\{L(f, \mathcal{P}) : \mathcal{P} \in \mathscr{P}[a, b]\} \leq \sup\{L(g, \mathcal{P}) : \mathcal{P} \in \mathscr{P}\} = L(g)$$

And so $L(f) \leq L(g)$. Also,

$$U(f) = \inf\{U(f, \mathcal{P}) : \mathcal{P} \in \mathscr{P}\} \leq \inf\{U(g, \mathcal{P}) : \mathcal{P} \in \mathscr{P}\} = U(g)$$

And thus $U(f) \leq U(g)$.

$\therefore$ If $f(x) \leq g(x)$, then $L(f) \leq L(g)$ and $U(f) \leq U(g)$. ■

5. Let f, g, h be bounded functions on $I := [a, b]$ such that $f(x) \leq g(x) \leq h(x)$ for all $x \in I$. Show that if f and h are Darboux integrable and if $\displaystyle\int_a^b f = \int_a^b h$, then g is also Darboux integrable with $\displaystyle\int_a^b g = \int_a^b f$.

Proof. Let $f, g, h : [a, b] \to \mathbb{R}$ be bounded functions such that $f(x) \leq g(x) \leq h(x)\ \forall\, x \in [a, b]$, and suppose f and h are both Darboux integrable, and $\displaystyle\int_a^b f = \int_a^b h$.

We want to show that g is also Darboux integrable and that $\displaystyle\int_a^b g = \int_a^b f$.

Since f and h are Darboux integrable, we know that $U(f) = L(f)$ and $U(h) = L(h)$. Thus by the theorem posed in *Problem 3*, we know that $U(f) \leq U(h)$ and $L(f) \leq L(h)$. We also know that $L(f) = U(f) = \displaystyle\int_a^b f$ and that $L(h) = U(h) = \displaystyle\int_a^b h = \int_a^b f$.

Again, by *Problem 3*, we have that $L(f) \leq L(g) \leq L(h)$ and $U(f) \leq U(g) \leq U(h)$. Thus we have that $\displaystyle\int_a^b f \leq L(g) \leq \int_a^b f$ and $\displaystyle\int_a^b f \leq U(g) \leq \int_a^b f$, which thus yields that $L(g) = \int_a^b f$, and that $U(g) = \int_a^b f$. Hence $L(g) = U(g) = \displaystyle\int_a^b f$.

$\therefore$ g is Darboux integrable and $\displaystyle\int_a^b g = \int_a^b f$. ■

6. Let f be defined on $[0, 2]$ by $f(x) := 1$ if $x \neq 1$ and $f(1) := 0$. Show that the Darboux integral exists and find its value.

Proof. Let $f(x) := \begin{cases} 1, & x \in [0,2] \setminus \{1\} \\ 0, & x = 1 \end{cases}$

Thus f is bounded on $[0, 2]$ Now, let $g : [0, 2] \to \mathbb{R}$ be given by $g(x) := 1$. Then since g is a constant function, we know that g is continuous on $[0, 2]$, and thus by *Theorem 7.2.7*, $g \in \mathcal{R}[0, 2]$.

So, $\displaystyle\int_0^2 1\ dx = 2 - 0 = 2$. And since $f(x) = g(x)\ \forall\ x \in [0, 2] \setminus \{1\}$, we have that $f \in \mathcal{R}[0, 2]$. Also, $\displaystyle\int_0^2 f = \int_0^2 g = 2$.

$\therefore$ By the *Equivalence Theorem*, since $f \in \mathcal{R}[0, 2]$, f is Darboux integrable and $\displaystyle\int_0^2 f = 2$.

■

7. **a.** Prove that if $g(x) := 0$ for $0 \leq x \leq \frac{1}{2}$ and $g(x) := 1$ for $\frac{1}{2} < x \leq 1$, then the Darboux integral of g on $[0, 1]$ is equal to $\frac{1}{2}$.

Proof. Let $g(x) := \begin{cases} 0, & 0 \leq x \leq \frac{1}{2} \\ 1, & \frac{1}{2} < x \leq 1 \end{cases}$. We want to show that the Darboux integral of g on $[0, 1]$ is equal to $\frac{1}{2}$.

Since g on the interval $\left[0, \frac{1}{2}\right]$ is a constant function, we know that g is continuous on that interval, and is thus Riemann integrable, whose evaluation yields $\displaystyle\int_0^{\frac{1}{2}} g = \int_0^{\frac{1}{2}} 0 = 0.$

Now, let $\varphi(x) := 1 \; \forall \; x \in \left[\frac{1}{2}, 1\right]$. Then we have that φ is a constant function and is thus continuous, and again by *Theorem 7.2.7*, φ is thus Riemann integrable, whose evaluation yields $\displaystyle\int_{\frac{1}{2}}^{1} 1 = 1 - \frac{1}{2} = \frac{1}{2}$. Thus, $g = \varphi$ except at $\frac{1}{2}$. Hence g is integrable on the interval $[0,1]$, and evaluates to $\displaystyle\int_0^1 g = \int_0^{\frac{1}{2}} g + \int_{\frac{1}{2}}^{1} g = \frac{1}{2}$.

Thus, by the *Equivalence Theorem*, we have that g is Darboux integrable. ∎

b. Does the conclusion hold if we change the value of g at the point $\frac{1}{2}$ to 13?

If we were to change g at the one point then Riemann integrability is not affected, thus if $g(\frac{1}{2}) = 13$, then g remains integrable on $[0,1]$ and $\displaystyle\int_0^1 g = \frac{1}{2}$. Thus, g is still Darboux integrable on $[0,1]$ with $\displaystyle\int_0^1 g = \frac{1}{2}$.

9. Let f_1 and f_2 be bounded functions on $[a,b]$. Show that $L(f_1) + L(f_2) \leq L(f_1 + f_2)$.

Proof. Consider the partitions $\mathcal{P}_1$ of f_1 on $[a,b]$, and $\mathcal{P}_2$ of f_2 on $[a,b]$, and let $\mathcal{P} := \mathcal{P}_1 \cup \mathcal{P}_2$ of $[a,b]$, where $\mathcal{P} := (x_0, x_1, \dots, x_n)$ such that

$$a = x_0 < x_1 < \cdots < x_n = b$$

and let $m_{i_1} := \inf\{f_1(x) : x \in [x_{i-1}, x_i]\}$, $m_{i_2} := \inf\{f_2(x) : x \in [x_{i-1}, x_i]\}$, and let $m_{i_3} := \inf\{f_1(x) + f_2(x) : x \in [x_{i-1}, x_i]\}$.

We note that $m_{i_1} + m_{i_2} \leq f_1(x) + f_2(x) \; \forall \; x \in [x_{i-1}, x_i]$.

Then we have the following:

$$\begin{aligned} L(f_1, \mathcal{P}) + L(f_2, \mathcal{P}) &= \sum_{i=1}^{n} m_{i_1}(x_i - x_{i-1}) + \sum_{i=1}^{n} m_{i_2}(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} (m_{i_1} + m_{i_2})(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{n} m_{i_3}(x_i - x_{i-1}), && \text{as was noted previously,} \\ &= L(f_1 + f_2, \mathcal{P}) \leq L(f_1 + f_2) \end{aligned}$$

Then we have that $\sup\{L(f_1,\mathcal{P})+L(f_2,\mathcal{P}) : \mathcal{P}\in\mathscr{P}[a,b]\} \leq L(f_1+f_2)$. Thus,

$$\begin{aligned} L(f_1)+L(f_2) &= \sup\{L(f_1,\mathcal{P}) : \mathcal{P}\in\mathscr{P}[a,b]\} + \sup\{L(f_2,\mathcal{P}) : \mathcal{P}\in\mathscr{P}[a,b]\} \\ &\leq \sup\{L(f_1+f_2,\mathcal{P}) : \mathcal{P}\in\mathscr{P}[a,b]\} \\ &= L(f_1+f_2) \end{aligned}$$

Thus we have that $L(f_1)+L(f_2)\leq L(f_1+f_2)$. ■

10. Give an example to show that strict inequality can hold in the preceding exercise.

Consider the functions $f_1, f_2 : [0,1]\to\mathbb{R}$ given by $f_1(x) := \begin{cases} 0, & x\in\mathbb{Q} \\ 1, & x\in\mathbb{R}\setminus\mathbb{Q} \end{cases}$, and $f_2(x) := \begin{cases} 1, & x\in\mathbb{Q} \\ 0, & x\in\mathbb{R}\setminus\mathbb{Q} \end{cases}$
Then we have that $L(f_1)=0$, and that $L(f_2)=0$, and thus $L(f_1)+L(f_2)=0$.

However, we now note that $(f_1+f_2)(x) = 1\ \forall\ x\in[0,1]$, and thus we have that $L(f_1+f_2)=U(f_1+f_2)=\int_0^1 1 = 1-0=1$. Thus we have that $0=L(f_1)+L(f_2)<L(f_1+f_2)=1$.

2. Let $f(x)=x^2$ on $[1,3.5]$.

(a) Find $L(f,P)$ and $U(f,P)$ when $P=\{1,2,3,3.5\}$.

Our terms are:

$$x_0 := 1, \quad x_1 := 2, \quad x_2 := 3, \quad x_3 := 3.5$$

and our subintervals are

$$I_1 := [1,2], \quad I_2 := [2,3], \quad I_3 := [3,3.5]$$

So for $L(f,\mathcal{P})$ we have

$$\begin{aligned}
L(f,\mathcal{P}) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} \inf\{f(x) : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} \inf\{x^2 : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) \\
&= \inf\{x^2 : x \in [1,2]\}(2-1) \\
&+ \inf\{x^2 : x \in [2,3]\}(3-2) \\
&+ \inf\{x^2 : x \in [3,3.5]\}(3.5-3) \\
&= 1 \cdot 1 + 4 \cdot 1 + 9 \cdot \frac{1}{2} \\
&= 1 + 4 + \frac{9}{2} \\
&= \frac{19}{2}
\end{aligned}$$

So $L(f,\mathcal{P}) = \frac{19}{2}$, and

$$\begin{aligned}
U(f,\mathcal{P}) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} \sup\{f(x) : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} \sup\{x^2 : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) \\
&= \sup\{x^2 : x \in [1,2]\}(2-1) \\
&+ \sup\{x^2 : x \in [2,3]\}(3-2) \\
&+ \sup\{x^2 : x \in [3,3.5]\}(3.5-3) \\
&= 4 \cdot 1 + 9 \cdot 1 + 12.25 \cdot \frac{1}{2} \\
&= 4 + 9 + 6.125 \\
&= 19.125
\end{aligned}$$

And so $U(f,\mathcal{P}) = 19.125$.

Thus $L(f,\mathcal{P}) = \frac{19}{2}$ and $U(f,\mathcal{P}) = 19.125$.

(b) Find $L(f,P)$ and $U(f,P)$ when $P = \{1, 1.5, 2, 2.5, 3, 3.5\}$.

Our terms are:

$$x_0 := 1, \quad x_1 := 1.5, \quad x_2 := 2, \quad x_3 := 2.5, \quad x_4 := 3, \quad x_5 := 3.5$$

and so our intervals are

$$I_1 := [1, 1.5], \quad I_2 := [1.5, 2], \quad I_3 := [2, 2.5], \quad I_4 := [2.5, 3], \quad I_5 := [3, 3.5]$$

So we have for $L(f, \mathcal{P})$:

$$\begin{aligned}
L(f, \mathcal{P}) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} \inf\{f(x) : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} \inf\{x^2 : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) \\
&= \inf\{x^2 : x \in [1, 1.5]\}(1.5 - 1) + \inf\{x^2 : x \in [1.5, 2]\}(2 - 1.5) \\
&+ \inf\{x^2 : x \in [2, 2.5]\}(2.5 - 2) + \inf\{x^2 : x \in [2.5, 3]\}(3 - 2.5) \\
&+ \inf\{x^2 : x \in [3, 3.5]\}(3.5 - 3) \\
&= 1 \cdot \frac{1}{2} + 2.25 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} + 6.25 \cdot \frac{1}{2} + 9 \cdot \frac{1}{2} \\
&= .5 + 1.125 + 2 + 3.125 + 4.5 \\
&= 11.25
\end{aligned}$$

thus $L(f, \mathcal{P}) = 11.25$ and

$$\begin{aligned}
U(f, \mathcal{P}) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} \sup\{f(x) : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} \sup\{x^2 : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) \\
&= \sup\{x^2 : x \in [1, 1.5]\}(1.5 - 1) + \sup\{x^2 : x \in [1.5, 2]\}(2 - 1.5) \\
&+ \sup\{x^2 : x \in [2, 2.5]\}(2.5 - 2) + \sup\{x^2 : x \in [2.5, 3]\}(3 - 2.5) \\
&+ \sup\{x^2 : x \in [3, 3.5]\}(3.5 - 3) \\
&= 2.25 \cdot 0.5 + 4 \cdot 0.5 + 6.25 \cdot 0.5 + 9 \cdot 0.5 + 12.25 \cdot 0.5 \\
&= 16.875
\end{aligned}$$

Thus $U(f, \mathcal{P}) = 16.875$.
Therefore $L(f, \mathcal{P}) = 11.25$ and $U(f, \mathcal{P}) = 16.875$.

3. Use upper and lower Darboux sums to evaluate the following integrals.

(a) $\displaystyle\int_1^3 (2x+3)\ dx$

Let $\mathcal{P}_n := \left(0, \frac{3}{n}, \frac{6}{n}, \ldots, \frac{3n-1}{n}, 3\right)$. Then $\Delta x_i := \frac{3}{n}$. Since $2x+3$ is increasing on $[1,3]$, we have that on $[x_{i-1}, x_i] = \left[\frac{3i-1}{n}, \frac{3i}{n}\right]$, $M_i =$ occurs at the right endpoint $= f(\frac{3i}{n}) = \frac{6i}{n} + 3$ and m_i occurs at the left endpoint, $f\left(\frac{3i-1}{n}\right) = \frac{6i-2}{n} + 3$.

So, we also note that in order to get the correct answer, we must calculate $\displaystyle\int_0^3 2x + 3\ dx - \int_0^1 2x+3\ dx$, and thus we choose $\mathcal{P}_1 := \left(0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right)$ with $\Delta x_{i_1} := \frac{1}{n}$, with $M_{i_1} := \frac{2i}{n} + 3$, and $m_{i_1} := \frac{2i-2}{n} + 3$, and thus:

$$
\begin{aligned}
U(f, \mathcal{P}_n) - U(f, \mathcal{P}_1) &= \sum_{i=1}^{n} M_i \Delta x_i - M_{i_1} \Delta x_{i_1} \\
&= \sum_{i=1}^{n} \left(\frac{6i}{n} + 3\right) \cdot \left(\frac{3}{n}\right) - \left[\left(\frac{2i}{n} + 3\right) \cdot \left(\frac{1}{n}\right)\right] \\
&= \sum_{i=1}^{n} \frac{18i}{n^2} + \frac{9}{n} - \frac{2i}{n^2} - \frac{3}{n} \\
&= \frac{18}{n^2} \sum_{i=1}^{n} i + \frac{9}{n} \sum_{i=1}^{n} 1 - \frac{2}{n^2} \sum_{i=1}^{n} i - \frac{3}{n} \sum_{i=1}^{n} 1 \\
&= \frac{18}{n^2} \cdot \frac{n(n+1)}{2} + \frac{9\cancel{n}}{\cancel{n}} - \frac{2}{n^2} \cdot \frac{n(n+1)}{2} - \frac{3\cancel{n}}{\cancel{n}} \\
&= \frac{18n^2 + 18n}{2n^2} + 9 - \frac{2n^2 + 2n}{2n^2} - 3 \\
&= \lim_{n \to \infty} \frac{18n^2 + 18n}{2n^2} + 9 - \frac{2n^2 + 2n}{2n^2} - 3 \\
&= 9 + 9 - 1 - 3 \\
&= 14 \\
&\geq U(f)
\end{aligned}
$$

and

$$
\begin{aligned}
L(f,\mathcal{P}_n) - L(f,\mathcal{P}_1) &= \sum_{i=1}^{n} m_i \Delta x_i - m_{i_1}\Delta x_{i_1} \\
&= \sum_{i=1}^{n} \left(\frac{6i-2}{n}+3\right)\cdot\left(\frac{3}{n}\right) - \left[\left(\frac{2i-2}{n}+3\right)\cdot\left(\frac{1}{n}\right)\right] \\
&= \sum_{i=1}^{n} \frac{18i-6}{n^2} + \frac{9}{n} - \frac{2i-2}{n^2} - \frac{3}{n} \\
&= \sum_{i=1}^{n} \frac{6(3i-1)}{n^2} + \frac{9}{n}\sum_{i=1}^{n} 1 - \frac{2}{n^2}\sum_{i=1}^{n}(i-1) - \frac{3}{n}\sum_{i=1}^{n} 1 \\
&= \frac{6}{n^2}\left(3\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\right) + \frac{9\cancel{n}}{\cancel{n}} - \frac{2}{n^2}\cdot\frac{(n-1)((n-1)+1)}{2} - \frac{3\cancel{n}}{\cancel{n}} \\
&= \frac{6}{n^2}\cdot\left(\frac{3n(n+1)}{2} - n\right) + 9 - \frac{2}{n^2}\cdot\frac{n^2-n}{2} - 3 \\
&= \frac{6}{n^2}\cdot\left(\frac{3n^2+3n}{2} - n\right) + 9 - \frac{2n^2-2n}{2n^2} - 3 \\
&= \frac{18n^2+18n}{2n^2} - \frac{6n}{n^2} + 9 - \frac{2n^2-2n}{2n^2} - 3 \\
&= \lim_{n\to\infty}\frac{18n^2+18n}{2n^2} - \lim_{n\to\infty}\frac{6}{n} + \lim_{n\to\infty} 9 - \lim_{n\to\infty}\frac{2n^2-2n}{2n^2} - \lim_{n\to\infty} 3 \\
&= 9 - 0 + 9 - 1 - 3 \\
&= 14 \\
&\le L(f)
\end{aligned}
$$

So,

$$14 \le L(f) \le U(f) \le 14$$

So $L(f) = U(f) = 14$.

(b) $\displaystyle\int_0^2 (x^2+1)\,dx$

Let $\varepsilon > 0$ be given, and let $\mathcal{P} := (0, \frac{2}{n}, \frac{4}{n}, \ldots, \frac{2n-1}{n}, 2)$, and we note that $\Delta x_i := \frac{2}{n}$.

Since f is increasing on $[0,2]$, then on $[x_{i-1}, x_i] = \left[\frac{2i-1}{n}, \frac{2i}{n}\right]$, M_i occurs at the right endpoint, and is thus $f\left(\frac{2i}{n}\right) = \frac{4i^2}{n^2} + 1$. Also, m_i occurs at the left endpoint and is thus $f\left(\frac{2i-1}{n}\right) = \left(\frac{4i^2-4i+1}{n^2} + 1\right)$.

$$\begin{aligned}
U(f, \mathcal{P}_n) &= \sum_{i=1}^{n} M_i \Delta x_i \\
&= \sum_{i=1}^{n} \left(\frac{4i^2}{n^2} + 1\right) \cdot \left(\frac{2}{n}\right) \\
&= \sum_{i=1}^{n} \frac{8i^2}{n^3} + \frac{2}{n} \\
&= \frac{8}{n^3} \ \sum_{i=1}^{n} i^2 + \frac{2}{n} \ \sum_{i=1}^{n} 1 \\
&= \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{2\cancel{n}}{\cancel{n}} \\
&= \frac{8}{n^3} \cdot \frac{2n^3+3n^2+n}{6} + 2 \\
&= \frac{16n^3+24n^2+8n}{6n^3} + 2 \\
&= \lim_{n\to\infty} \frac{16n^3+24n^2+8n}{6n^3} + 2 \\
&= \frac{16}{6} + 2 \\
&= \frac{8}{3} + 2 \\
&= \frac{14}{3} \\
&\geq U(f)
\end{aligned}$$

and

$$
\begin{aligned}
L(f, \mathcal{P}_n) &= \sum_{i=1}^{n} m_i \Delta x_i \\
&= \sum_{i=1}^{n} \left(\frac{4i^2 - 4i + 1}{n^2} + 1 \right) \cdot \left(\frac{2}{n} \right) \\
&= \sum_{i=1}^{n} \frac{8i^2 - 8i + 2}{n^3} + \frac{2}{n} \\
&= \sum_{i=1}^{n} \frac{2(2i-1)^2}{n^3} + \frac{2}{n} \sum_{i=1}^{n} 1 \\
&= \frac{8}{3} - \frac{2}{3n^2} + \frac{2\cancel{n}}{\cancel{n}} \\
&= \lim_{n \to \infty} \frac{8}{3} - \frac{2}{3n^2} + 2 \\
&= \frac{8}{3} - 0 + 2 \\
&= \frac{8}{3} + 2 \\
&= \frac{14}{3} \\
&\leq L(f)
\end{aligned}
$$

So $\frac{14}{3} \leq L(f) \leq U(f) \leq \frac{14}{3}$. So $L(f) = U(f) = \frac{14}{3}$

4. (a) Prove that if $f, g : [a, b] \to \mathbb{R}$ are bounded, then $U(f+g, \mathcal{P}) \leq U(f, \mathcal{P}) + U(g, \mathcal{P})$ for every partition $\mathcal{P}$ of $[a, b]$.

Proof. Since f and g are bounded, we know that $\sup(f+g) \leq \sup(f) + \sup(g)$. Then $M_{f+g,i} := \sup\{f + g : x \in [x_{i-1}, x_i]\}$. And so $M_{f+g,i} \leq M_{f,i} + M_{g,i}$, and thus

$$U(f+g, \mathcal{P}) := \sum_{i=1}^{n} M_{f+g,i} \Delta x_i \leq \sum_{i=1}^{n} M_{f,i} \Delta x_i + \sum_{i=1}^{n} M_{g,i} \Delta x_i = U(f, \mathcal{P}) + U(g, \mathcal{P})$$

■

(b) Find examples of bounded functions $f, g : [a, b] \to \mathbb{R}$ such that $U(f+g, \mathcal{P}) < U(f, \mathcal{P}) + U(g, \mathcal{P})$ for some partition of $[a, b]$.

Consider the functions $f, g : [a, b] \to \mathbb{R}$ given by $f(x) := \begin{cases} 1, & x \in \mathbb{R} \setminus \mathbb{Q} \\ 0, & x \in \mathbb{Q} \end{cases}$ and $g(x) := \begin{cases} -1, & x \in \mathbb{R} \setminus \mathbb{Q} \\ 0, & x \in \mathbb{Q} \end{cases}$

Thus, we have that $(f+g)(x) = 0 \; \forall \; x$, and thus $U(f+g) = 0$. However, we note that $U(f, \mathcal{P}) + U(g, \mathcal{P}) = 1 + 0 = 1$. Thus, $0 = U(f+g, \mathcal{P}) < U(f, \mathcal{P}) + U(g, \mathcal{P}) = 1$.

5. Prove or justify, if true or provide a counterexample, if false.

(a) Let f be bounded on $[a, b]$. The upper and lower sums for f form a bounded set.

This is a true statement.

Proof. Since f is bounded, we know that $\exists\, m, M$ s.t. $m \leq f(x) \leq M \;\forall\, x \in [a, b]$. By the definitions of $L(f, \mathcal{P})$, and $U(f, \mathcal{P})$, we have

$$L(f, \mathcal{P}) := \sum_{i=1}^{n} \inf(f) \cdot \Delta x_i \quad \text{and} \quad U(f, \mathcal{P}) := \sum_{i=1}^{n} \sup(f) \cdot \Delta x_i$$

This yields that

$$m(b-a) \leq L(f, \mathcal{P}) \leq U(f, \mathcal{P}) \leq M(b-a)$$

So $L(f, \mathcal{P})$, and $U(f, \mathcal{P})$ are bounded. ∎

(b) Let f be bounded on $[a, b]$. $f \in \mathcal{R}[a, b]$ if and only if its lower and upper sums are equal.

This is a false statement. Consider the function and partition given in *Problem 1 (a)*: Let $f(x) := |x|$ for $-1 \leq x \leq 2$. Calculate $L(f; P)$ and $U(f, P)$ for the following partition: $\mathcal{P}_1 := (-1, 0, 1, 2)$

Our terms are:

$$x_0 := -1, \quad x_1 := 0, \quad x_2 := 1, \quad x_3 := 2$$

and our intervals are:

$$I_1 := [-1, 0], \quad I_2 := [0, 1], \quad I_3 := [1, 2]$$

thus $L(f, \mathcal{P}_1)$ is:

$$\begin{aligned}
L(f, \mathcal{P}_1) &= \sum_{i=1}^{n} m_i (x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} \left(\inf\{f(x) : x \in [x_{i-1}, x_i]\}\right)(x_i - x_{i-1}) \\
&= (\inf\{|x| : x \in [-1, 0]\})(0 - (-1)) \\
&+ (\inf\{|x| : x \in [0, 1]\})(1 - 0) \\
&+ (\inf\{|x| : x \in [1, 2]\})(2 - 1) \\
&= 0 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 \\
&= 0 + 0 + 1 \\
&= 1
\end{aligned}$$

and

$$
\begin{aligned}
U(f,\mathcal{P}_1) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} \sup\{f(x) : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} \sup\{|x| : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) \\
&= (\sup\{|x| : x \in [-1,0]\})(0-(-1)) \\
&+ (\sup\{|x| : x \in [0,1]\})(1-0) \\
&+ (\sup\{|x| : x \in [1,2]\})(2-1) \\
&= 1 \cdot 1 + 1 \cdot 1 + 2 \cdot 1 \\
&= 1 + 1 + 2 \\
&= 4
\end{aligned}
$$

So, $L(f,\mathcal{P}_1) = 1$ and $U(f,\mathcal{P}_1) = 4$

(c) Let f be bounded on $[a,b]$. If P and Q are partitions of $[a,b]$, then $L(f,P) \leq U(f,Q)$.

This is a true statement by *Lemma 7.4.3*:

Lemma. Let $f : I \to \mathbb{R}$ be bounded. If $\mathcal{P}_1, \mathcal{P}_2$ are any two partitions of I, then $L(f;\mathcal{P}_1) \leq U(f;\mathcal{P}_2)$.

(d) When $\displaystyle\int_a^b f(x)\,dx$ exists, it is the unique number that lies between $L(f,P)$ and $U(f,P)$ for all partitions P of $[a,b]$.

This is true since if $f \in \mathcal{R}[a,b]$, and if we let $I := [a,b]$, then f is Darboux integrable, by the *Equivalence Theorem.* Then, by the definition of the Darboux integral, $U(f) = \inf\{U(f,\mathcal{P}) : \mathcal{P} \in \mathscr{P}(I)\}$ and $L(f) := \sup\{L(f,\mathcal{P}) : \mathcal{P} \in \mathscr{P}(I)\}$, and by *Theorem 7.1.1*:

Theorem. If $f \in \mathcal{R}[a,b]$, then the value of the integral is uniquely determined.

, we have that the value L of $\displaystyle\int_a^b f(x)\,dx = L$ is uniquely determined for all partitions $\mathcal{P} \in \mathscr{P}(I)$.

That is, this statement is a combination of *Theorem 7.4.1* and *Theorem 7.1.2.*

(e) Let f be bounded on $[a,b]$. Then $L(f,P) \leq \displaystyle\int_a^b f(x)\,dx \leq U(f,P)$.

This is false. Consider the Dirichlet function on the interval $[0,1]$: $f : [0,1] \to \mathbb{R}$

given by $f(x) := \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

Then we have that $L(f, \mathcal{P}) = 0$ and $U(f, \mathcal{P}) := 1$, so we have that $L(f, \mathcal{P}) \leq U(f, \mathcal{P})$, but we know that the Dirichlet function is not integrable. Thus we have that this is a false statement.

(f) If $f \in \mathcal{R}[a, b]$, then for all $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that $L(f, P) > U(f, P) - \varepsilon$.

This is a true statement.

Proof. Let $f \in \mathcal{R}[a, b]$. Recall the *Equivalence Theorem*:

Theorem. Equivalence Theorem A function f on $I = [a, b]$ is Darboux integrable if and only if it is Riemann integrable.

Thus by the *Equivalence Theorem*, f is also Darboux integrable.

Also, recall the *Integrability Criterion*:

Theorem. Integrability Criterion Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ be a bounded function on I. Then f is Darboux integrable on I if and only if for each $\varepsilon > 0$ there is a partition $\mathcal{P}_\varepsilon$ of I such that

$$U(f; \mathcal{P}_\varepsilon) - L(f; \mathcal{P}_\varepsilon) < \varepsilon$$

Note that we can rewrite the inequality as follows:

$$\begin{aligned} U(f, \mathcal{P}_\varepsilon) - L(f, \mathcal{P}_\varepsilon) < \varepsilon &\equiv -L(f, \mathcal{P}_\varepsilon) < \varepsilon - U(f, \mathcal{P}_\varepsilon) \\ &\equiv L(f, \mathcal{P}\varepsilon) > U(f, \mathcal{P}_\varepsilon) - \varepsilon \end{aligned}$$

Thus by the *Integrability Criterion*, we have that $L(f, \mathcal{P}) > U(f, \mathcal{P}) - \varepsilon$. ∎