Real Analysis II Homework 1

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Section 7.1 - The Riemann Integral

1. **1.** If $I := [0, 4]$, calculate the norms of the following partitions:

c. $\mathcal{P}_3 := (0, 1, 1.5, 2, 3.4, 4)$
$||\mathcal{P}_3|| = 1.4$

d. $\mathcal{P}_4 := (0, .5, 2.5, 3.5, 4)$
$||\mathcal{P}_4|| = 2$

2. If $f(x) := x^2$ for $x \in [0, 4]$, calculate the following Riemann sums, where $\dot{\mathcal{P}}_i$ has the same partition points as in Exercise 1, and the tags are selected as indicated.
$\dot{\mathcal{P}}_1 := (0, 1, 2, 4)$
$\dot{\mathcal{P}}_2 := (0, 2, 3, 4)$

(a) $\dot{\mathcal{P}}_1$ with the tags at the left endpoints of the subintervals.
The subintervals are:

$$I_1 := [0, 1],\ I_2 := [1, 2],\ I_3 := [2, 4]$$

So the tags are:

$$t_1 := 0,\ t_2 := 1,\ t_3 := 2$$

$$\begin{aligned}
S(f, \dot{\mathcal{P}}_1) &= \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \\
&= f(0)(x_1 - x_0) + f(1)(x_2 - x_1) + f(2)(x_3 - x_2) \\
&= 0^2(1 - 0) + 1^2(2 - 1) + 2^2(4 - 2) \\
&= 1 + 8 \\
&= 9
\end{aligned}$$

(b) $\dot{\mathcal{P}}_1$ with the tags at the right endpoints of the subintervals.
The subintervals are

$$I_1 := [0, 1],\ I_2 := [1, 2],\ I_3 := [2, 4]$$

So the tags are:

$$t_1 := 1,\ t_2 := 2,\ t_3 := 4$$

$$
\begin{aligned}
S(f, \dot{\mathcal{P}}_1) &= \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \\
&= f(1)(x_1 - x_0) + f(2)(x_2 - x_1) + f(4)(x_3 - x_2) \\
&= 1^2(1-0) + 2^2(2-1) + 4^2(4-2) \\
&= 1 + 4 + 32 \\
&= 37
\end{aligned}
$$

(c) $\dot{\mathcal{P}}_2$ with the tags at the left endpoints of the subintervals.
The subintervals are:

$$I_1 := [0,2],\ I_2 := [2,3],\ I_3 := [3,4]$$

So the tags are:

$$t_1 := 0,\ t_2 := 2,\ t_3 := 3$$

$$
\begin{aligned}
S(f, \dot{\mathcal{P}}_2) &= \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \\
&= f(0)(x_1 - x_0) + f(2)(x_2 - x_1) + f(3)(x_3 - x_2) \\
&= 0^2(2-0) + 2^2(3-2) + 3^2(4-3) \\
&= 4 + 9 \\
&= 13
\end{aligned}
$$

(d) $\dot{\mathcal{P}}_2$ with the tags at the right endpoints of the subintervals.
So the subintervals are:

$$I_1 := [0,2],\ I_2 := [2,3],\ I_3 := [3,4]$$

So the tags are:

$$t_1 := 2,\ t_2 := 3,\ t_3 := 4$$

$$
\begin{aligned}
S(f, \dot{\mathcal{P}}_2) &= \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \\
&= f(2)(2-0) + f(3)(3-2) + f(4)(4-3) \\
&= 2^2(2) + 3^2(1) + 4^2(1) \\
&= 8 + 9 + 16 \\
&= 33
\end{aligned}
$$

6. (a) Let $f(x) := 2$ if $0 \le x < 1$ and $f(x) := 1$ if $1 \le x \le 2$. Show that $f \in \mathcal{R}[0,2]$ and evaluate its integral.

We estimate by the graph of f that the integral of f is 3. We must now show by the definition of the integral that the integral of f is 3.

Proof. Let $\dot{\mathcal{P}}$ be a tagged partition of $[0,2]$. Let $\dot{\mathcal{P}}_1 \subseteq \dot{\mathcal{P}}$ with tags in $[0,1]$, and let $\dot{\mathcal{P}}_2 \subseteq \dot{\mathcal{P}}$ with tags in $[1,2]$.

We know that

$$[0, 1 - ||\dot{\mathcal{P}}||] \subseteq U_1 \subseteq [0, 1 + ||\dot{\mathcal{P}}||] \quad (1)$$

and

$$[1 + ||\dot{\mathcal{P}}||, 2] \subseteq U_2 \subseteq [1 - ||\dot{\mathcal{P}}||, 2] \quad (2)$$

where U_1 and U_2 are the union of the subintervals $\dot{\mathcal{P}}_1$ and $\dot{\mathcal{P}}_2$, respectively.

Now, we can calculate $S(f;\dot{\mathcal{P}}_1)$ and $S(f;\dot{\mathcal{P}}_2)$.

$$\begin{aligned}
S(f;\dot{\mathcal{P}}_1) &= \sum_{I_i \in \dot{\mathcal{P}}_1} f(t_i)(x_i - x_{i-1}) \\
&= \sum_{I_i \in \dot{\mathcal{P}}_1} 2(x_i - x_{i-1}) \\
&(I_i \in \dot{\mathcal{P}}_1 \implies I_i \subseteq [0,1] \text{ where the function value is } 2) \\
&= 2 \sum_{I_i \in \dot{\mathcal{P}}_1} (x_i - x_{i-1}) \\
&\in [2(1 - ||\dot{\mathcal{P}}||), 2(1 + ||\dot{\mathcal{P}}||)] = [2 - 2||\dot{\mathcal{P}}||, 2 + 2||\dot{\mathcal{P}}||]
\end{aligned}$$

(Because of (1) we know that the range of the subinterval lengths in $\dot{\mathcal{P}}_1$)

$$\begin{aligned}
S(f;\dot{\mathcal{P}}_2) &= \sum_{I_i \in \dot{\mathcal{P}}_2} f(t_i)(x_i - x_{i-1}) \\
&= \sum_{I_i \in \dot{\mathcal{P}}_2} 1(x_i - x_{i-1}) \\
&(I_i \in \dot{\mathcal{P}}_2 \implies I_i \subseteq [1,2] \text{ where the function value is } 1) \\
&= \sum_{I_i \in \dot{\mathcal{P}}_2} (x_i - x_{i-1}) \\
&\in [1 - ||\dot{\mathcal{P}}||, 1 + ||\dot{\mathcal{P}}||]
\end{aligned}$$

(Because of (2), we know the range of the subinterval lengths in $\dot{\mathcal{P}}_2$)

Therefore,

$$S(f;\dot{\mathcal{P}}) = S(f;\dot{\mathcal{P}}_1) + S(f;\dot{\mathcal{P}}_2) \in [3(1 - ||\dot{\mathcal{P}}||), 3(1 + ||\dot{\mathcal{P}}||)]$$

$$\Updownarrow$$

$$3 - 3||\dot{\mathcal{P}}|| \leq S(f;\dot{\mathcal{P}}) \leq 3 + 3||\dot{\mathcal{P}}||$$

$$\Updownarrow$$

$$|S(f;\dot{\mathcal{P}}) - 3| \leq 3||\dot{\mathcal{P}}||$$

For arbitrary $\varepsilon > 0$ we can pick a tagged partition $\dot{\mathcal{P}}$ such that

$$||\dot{\mathcal{P}}|| < \frac{\varepsilon}{3}$$

Thus $f \in \mathcal{R}[0,2]$. ■

8. If $f \in \mathcal{R}[a,b]$ and $|f(x)| \leq M$ for all $x \in [a,b]$, show that $\left|\int_a^b f\right| \leq M(b-a)$.
Note that

$$-M \leq |f(x)| \leq M, \ \forall \ x \in [a,b]$$

By *Theorem 7.1.5 c*, and since every constant function on $[a,b]$ is in $\mathcal{R}[a,b]$, we have that

$$-M(b-a) \leq \int_a^b (-M) \leq \int_a^b f \leq \int_a^b M = M(b-a)$$

Therefore,

$$\left|\int_a^b f\right| \leq M(b-a)$$

12. Consider the Dirichlet function, introduced in Example 5.1.6(g), defined by $f(x) := 1$ for $x \in [0,1]$ rational and $f(x) := 0$ for $x \in [0,1]$ irrational. Use the preceding exercise to show that f is *not* Riemann integrable on $[0,1]$.

Let

$$\dot{\mathcal{P}}_n := \left\{\left[\frac{i-1}{n}, \frac{i}{n}\right], \frac{i}{n}\right\}_{i=1}^{n}, \ n \geq 1$$

Then $||\dot{\mathcal{P}}_n|| = \frac{1}{n} \to 0$ as $n \to \infty$.
Then,

$$S(f;\dot{\mathcal{P}}_n) := \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{i}{n} - \frac{i-1}{n}\right) = \sum_{i=1}^{n} 1 \cdot \frac{i}{n} = 1$$

because $\frac{i}{n}$ is rational.

Let

$$\dot{\mathcal{Q}}_n := \left\{\left[\frac{i-1}{n}, \frac{i}{n}\right], \alpha_i\right\}_{i=1}^{n}, \ n \geq 1$$

where α_i is an irrational number in the interval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$, for $i = 1, 2, \ldots, n$.
Then $||\dot{\mathcal{Q}}_n|| = \frac{1}{n} \to 0$ as $n \to \infty$.
Then,

$$S(f;\dot{\mathcal{Q}}_n) := \sum_{i=1}^{n} f(\alpha_i)\left(\frac{i}{n} - \frac{i-1}{n}\right) = \sum_{i=1}^{n} 0 \cdot \frac{i}{n} = 0$$

because α_i is irrational.

Therefore,

$$\lim_n S(f;\dot{\mathcal{P}}_n) = 1 \neq 0 = \lim_n S(f;\dot{\mathcal{Q}}_n)$$

By the definition of a Riemann integrable function, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all tagged partitions $\dot{\mathcal{P}}$ with $||\dot{\mathcal{P}}|| < \delta$ we have

$$\left|S(f;\dot{\mathcal{P}}) - \int_a^b f\right| < \frac{\varepsilon}{2}$$

Because $||\dot{\mathcal{P}}_n|| \to 0$, there exists $n_1 \in \mathbb{N}$ such that

$$n > n_1 \implies ||\dot{\mathcal{P}}_n|| < \delta$$

Similarly, because $||\dot{\mathcal{Q}}_n|| \to 0$, there exists $n_2 \in \mathbb{N}$ such that

$$n > n_2 \implies ||\dot{\mathcal{Q}}_n|| < \delta$$

Let $n_0 := \max\{n_1, n_2\}$. Then for all $n > n_0$ we have that

$$||\dot{\mathcal{P}}_n|| < \delta \ \& \ ||\dot{\mathcal{Q}}_n|| < \delta$$

so we have

$$\left|S(f;\dot{\mathcal{P}}_n) - \int_a^b f\right| < \frac{\varepsilon}{2} \ \& \ \left|S(f;\dot{\mathcal{Q}}_n) - \int_a^b f\right| < \frac{\varepsilon}{2}$$

Therefore, for all $n > n_0$,

$$\begin{aligned}\left|S(f;\dot{\mathcal{P}}_n) - S(f;\dot{\mathcal{Q}}_n)\right| &< \left|S(f;\dot{\mathcal{P}}_n) - \int_a^b f\right| + \left|S(f;\dot{\mathcal{Q}}_n) - \int_a^b f\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

By the definition of the limit of a sequence,

$$\lim_n \left[S(f;\dot{\mathcal{P}}_n) - S(f;\dot{\mathcal{Q}}_n)\right] = 0,$$

that is,

$$\lim_n S(f;\dot{\mathcal{P}}_n) = \lim_n S(f;\dot{\mathcal{Q}}_n)$$

which is a contradiction. Therefore $f \notin \mathcal{R}[a,b]$, and hence the Dirichlet function is not Riemann integrable.

2. **8.** Suppose that f is continuous on $[a,b]$, that $f(x) \geq 0$ for all $x \in [a,b]$ and that $\int_a^b f = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$.

Proof. Suppose there exists $c \in [a,b]$ such that $f(c) > 0$. Since f is continuous, there exists $\delta > 0$ such that $f(x) > \frac{1}{2}f(c)$ for $x \in (c-\delta, c+\delta) \subseteq [a,b]$. Then

$$\int_a^b f \geq \int_{c-\delta}^{c+\delta} f \geq \frac{1}{2}f(c) \cdot 2\delta > 0$$

which contradicts the fact that $\int_a^b f = 0$. If $c = a$, then there exists $\delta > 0$ such that $f(x) > 0$ for $x \in [a, a+\delta)$, and thus the same contradiction is present. The same applies for the case in which $a = b$. Therefore we have that $f(x) = 0$, $\forall\, x \in [a,b]$. ■

9. Show that the continuity hypothesis in the preceding exercise cannot be dropped.

Consider the function $f : [0, 1] \to \mathbb{R}$ given by

$$f(x) := \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

Then, f has a discontinuity at the point $x = 0$ and $\int_0^1 f = 0$, but f is not zero everywhere. Therefore, continuity is a necessary part of the hypothesis.

10. If f and g are continuous on $[a, b]$ and if $\int_a^b f = \int_a^b g$, prove that there exists $c \in [a, b]$ such that $f(c) = g(c)$.

Proof. Let f and g be continuous functions on $[a, b]$ such that

$$\int_a^b f = \int_a^b g$$

Define $h : [a, b] \to \mathbb{R}$ as $h := f - g$. Then, h is continuous as a difference of continuous functions and

$$\int_a^b h = \int_a^b (f - g) = \int_a^b f - \int_a^b g = 0$$

Suppose that there exists $c \in [a, b]$ such that $h(c) = 0$ since $(f(c) = g(c))$. Then, since h is continuous, it follows that $h(x) > 0,\ \forall\ x \in [a, b]$. or $h(x) < 0,\ \forall\ x \in [a, b]$ (recall *Bolzano's Theorem*).

Suppose $h(x) > 0,\ \forall\ x \in [a, b]$. Then because h is a continuous function on a segment, by the *Maximum-Minimum Theorem* there exists $m > 0$ such that

$$h(x) \geq m > 0,\ \forall\ x \in [a, b]$$

Then we have

$$\int_a^b h \geq \int_a^b m = m(b - a) > 0$$

This is a contradiction with the fact that $\int_a^b h = 0$.

Now, for the case in which $h(x) < 0,\ \forall\ x \in [a, b]$, by the *Maximum-Minimum Theorem* we know that there exists $M < 0$ such that

$$h(x) \leq M < 0\ \forall\ x \in [a, b]$$

and thus

$$\int_a^b h \leq \int_a^b M \leq M(b - a) < 0$$

which again yields a contradiction.

Therefore, there exists $c \in [a, b]$ such that $h(c) = 0$, that is, $f(c) = g(c)$. ∎

13. Give an example of a function $f : [a, b] \to \mathbb{R}$ that is in $\mathcal{R}[c, b]$ for every $c \in (a, b)$ but which is not in $\mathcal{R}[a, b]$.

Define a function f on $[0, 1]$ by

$$f(x) := \begin{cases} \frac{1}{x}, & x \in (0,1] \\ 1, & x = 0 \end{cases}$$

For every $c > 0$, $f \in \mathcal{R}[c, 1]$ because f is continuous on $[c, 1]$.

Now, let's show that f isn't Riemann integrable on $[0, 1]$.

Define a tagged partition to be

$$\dot{\mathcal{P}} := \left\{ \left[\frac{i-1}{n}, \frac{i}{n} \right], \frac{i}{n} \right\}_{i=1}^{n}$$

Then

$$\begin{aligned} S(f; \dot{\mathcal{P}}) &= \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{i}{n} - \frac{i-1}{n}\right) \\ &= \sum_{i=1}^{n} \frac{1}{\frac{i}{n}} \cdot \frac{1}{n} \\ &= \sum_{i=1}^{n} \frac{1}{i} \end{aligned}$$

As $n \to \infty$, $S(f; \dot{\mathcal{P}})$ diverges (since it is a harmonic series). Thus, f is not Riemann integrable on $[0, 1]$.

3. Use the right-endpoint Riemann sums to evaluate the following integrals:

(a) $\int_2^5 (3x-1)dx$

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\Delta x_i &= \sum_{i=1}^{n} f(a+i\Delta x)\Delta x \\
&= \sum_{i=1}^{n} f\left(2+i\left(\frac{3}{n}\right)\right)\cdot\left(\frac{3}{n}\right) \\
&= \sum_{i=1}^{n} \left(3\cdot\left(2+\frac{3i}{n}\right)-1\right)\cdot\left(\frac{3}{n}\right) \\
&= \sum_{i=1}^{n} \left(6+\frac{9i}{n}-1\right)\left(\frac{3}{n}\right) \\
&= \sum_{i=1}^{n} \left(5+\frac{9i}{n}\right)\left(\frac{3}{n}\right) \\
&= \sum_{i=1}^{n} \frac{15}{n}+\frac{27i}{n^2} \\
&= \frac{15}{n}\sum_{i=1}^{n} 1+\frac{27}{n^2}\sum_{i=1}^{n} i \\
&= \frac{15}{\cancel{n}}\cdot\cancel{n}+\frac{27}{n^2}\cdot\frac{n(n+1)}{2} \\
&= 15+\frac{27}{n^2}\cdot\frac{n^2+n}{2} \\
&= 15+\frac{27n^2+27n}{2n^2} \\
\int_2^5 (3x-1)dx &= \lim_{n\to\infty} 15+\frac{27n^2+27n}{2n^2} \\
&= 15+\frac{27}{2} \\
&= 28.5
\end{aligned}$$

(b) $\int_0^4 (x^2+2x)dx$

$$\begin{aligned}
\int_0^4 (x^2+2x)dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i)\Delta x \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(a+i\Delta x\right)\Delta x \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(0+i\left(\frac{4}{n}\right)\right)\cdot\left(\frac{4}{n}\right) \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} \left[\left(\frac{4i}{n}\right)^2+2\left(\frac{4i}{n}\right)\cdot\right]\left(\frac{4}{n}\right) \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} \left(\frac{16i^2}{n^2}+\frac{8i}{n}\right)\cdot\left(\frac{4}{n}\right) \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} \left(\frac{64i^2}{n^3}+\frac{32i}{n^2}\right) \\
&= \lim_{n\to\infty}\left(\frac{64}{n^3}\ \sum_{i=1}^{n} i^2+\frac{32}{n^2}\ \sum_{i=1}^{n} i\right) \\
&= \lim_{n\to\infty}\left(\frac{64}{n^3}\cdot\frac{n(n+1)(n+2)}{6}+\frac{32}{n^2}\cdot\frac{n(n+1)}{2}\right) \\
&= \lim_{n\to\infty}\left(\frac{64n^3+192n^2+128n}{6n^3}+\frac{32n^2+32n}{2n^2}\right) \\
&= \frac{64}{6}+16 \\
&\approx 26.6667
\end{aligned}$$

(c) $\int_0^2 (2x^3 + x)dx$

$$\begin{aligned}
\int_0^2 (2x^3 + x)dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f(a + i\Delta x)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(2\left(\frac{2i}{n}\right)^3 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{32i^3}{n^4} + \frac{4i}{n^2}\right) \\
&= \lim_{n\to\infty} \left[\frac{32}{n^4} \sum_{i=1}^{n} i^3 + \frac{4}{n^2} \sum_{i=1}^{n} i\right] \\
&= \lim_{n\to\infty} \left[\frac{32}{n^4} \cdot \frac{n^2(n+1)^2}{4} + \frac{4}{n^2} \cdot \frac{n(n+1)}{2}\right] \\
&= \lim_{n\to\infty} \left[\frac{32n^4 + 64n^3 + 32n^2}{4n^4} + \frac{4n^2 + 4n}{2n^2}\right] \\
&= 8 + 2 \\
&= 10
\end{aligned}$$

4. Express each of the following as a definite integral. Then use calculus to evaluate the integral.

(a) $\lim_{|P|\to 0} \sum_{i=1}^{n} \left(\frac{3}{w_i^2}\right) \Delta x_i$ where P is a partition of $[1, 3]$.

$$\begin{aligned}
\lim_{|P|\to 0} \sum_{i=1}^{n} \left(\frac{3}{w_i^2}\right) \Delta x_i &= \int_1^3 x dx \\
&= \left.\frac{x^2}{2}\right|_1^3 \\
&= \frac{3^2}{2} - \frac{1^2}{2} \\
&= \frac{9}{2} - \frac{1}{2} \\
&= \frac{8}{2} \\
&= 4
\end{aligned}$$

(b) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(3+\frac{2i}{n}\right)^2 \cdot \left(\frac{2}{n}\right)$

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} \left(3+\frac{2i}{n}\right)^2 \cdot \left(\frac{2}{n}\right) &= \lim_{n\to\infty} \sum_{i=1}^{n} (a+i\Delta x)^2 \cdot (\Delta x) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(a+i\left(\frac{b-a}{n}\right)\right)^2 \cdot \left(\frac{b-a}{n}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(3+i\left(\frac{2}{n}\right)\right)^2 \cdot \left(\frac{2}{n}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(3+i\left(\frac{5-3}{n}\right)\right)^2 \cdot \left(\frac{5-3}{n}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i) \cdot \Delta x \\
&= \int_3^5 f(t_i)dx \\
&= \int_3^5 x^2 dx \\
&= \left.\frac{x^3}{3}\right|_3^5 \\
&= \frac{5^3}{3} - \frac{3^3}{3} \\
&= \frac{125}{3} - \frac{27}{3} \\
&= \frac{125}{3} - 9 \\
&\approx 32.66667
\end{aligned}$$

(c) $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \left(1 + \frac{4(i-1)}{n}\right)^5 \cdot \left(\frac{4}{n}\right)$

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{4(i-1)}{n}\right)^5 \cdot \left(\frac{4}{n}\right) &= \lim_{n\to\infty} \sum_{i=1}^{n} (a + i\Delta x)^5 \cdot (\Delta x) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(a + i\left(\frac{b-a}{n}\right)\right)^5 \cdot \left(\frac{b-a}{n}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{4i-4}{n}\right)^5 \cdot \left(\frac{4}{n}\right) \\
&= \int_1^5 f(t_i)dx \\
&= \int_1^5 x^5 dx \\
&= \frac{x^6}{6}\bigg|_1^5 \\
&= \frac{5^6}{6} - \frac{1^6}{6} \\
&= \frac{15625}{6} - \frac{1}{6} \\
&= \frac{15624}{6} \\
&= \frac{7812}{3} \\
&= 2604
\end{aligned}$$

(d) $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{i^2}{n^3}$

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^3} &= \lim_{n\to\infty} \sum_{i=1}^{n} (a + i\Delta x)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(a + i\left(\frac{b-a}{n}\right)\right) \cdot \left(\frac{b-a}{n}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(0 + i\left(\frac{1-0}{n}\right)\right)^2 \cdot \left(\frac{1-0}{n}\right) \\
&= \int_0^1 f(t_i)dx \\
&= \int_0^1 x^2 dx \\
&= \frac{x^3}{3}\bigg|_0^1 \\
&= \frac{1^3}{3} - \frac{0^3}{3} \\
&= \frac{1}{3}
\end{aligned}$$

(e) Show that $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{n}{n^2+i^2} = \frac{\pi}{4}$

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{n^2+i^2} &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{n^2+i^2} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{1+(\frac{i}{n})^2} \cdot \frac{1}{n} \\
&= \int_0^1 \frac{1}{x^2} dx \\
&= \arctan(x)|_0^1 \\
&= \arctan(1) - \arctan(0) \\
&= \frac{\pi}{4} - 0 \\
&= \frac{\pi}{4}
\end{aligned}$$

5. Give examples of functions $f : [0,1] \to \mathbb{R}$ such that

(a) $f \notin \mathcal{R}[0,1]$, but $|f|$ and f^2 are both in $\mathcal{R}[0,1]$.

Consider the function $f : [0,1] \to \mathbb{R}$ given by $f(x) := \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$, and let

$P := \{x_0, x_1, \ldots, x_n\}$ be any partition of $[0,1]$. Then $M_i = 1$ and $m_i = -1$ for all $i = 1, 2, \ldots, n$. thus $U(f,P) = 1$ and $L(f,P) = -1$ for all P. Thus $U(f) = 1$ and $L(f) = -1$. Thus f is not integrable.

However, $|f|(x) = 1$ for all $x \in [0,1]$. Since $|f|$ is a continuous function $|f|$ is integrable on $[0,1]$, and also since $f^2(x) = 1$ is also a continuous function, we have that f^2 is also integrable on $[0,1]$.

(b) f is bounded, but $f \notin \mathcal{R}[0,1]$.

Consider the function $f : [0,1] \to \mathbb{R}$ given by $f(x) := \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Let P be any partition of $[0,1]$, then

$$U(P; f) := \sum M_i \Delta x_i = \sum 1 \Delta x_i = b - a$$

and

$$L(P; f) = \sum m_i \Delta x_i = \sum 0 \Delta x_i = 0(b-a) = 0$$

Hence

$$\overline{\int_0^1} f dx = b - a \neq 0 = \underline{\int_0^1} f dx$$

Hence f is not Riemann integrable.

(c) $f \in \mathcal{R}[0,1]$ and f is not monotone.

Consider the function $f : [0,1] \to \mathbb{R}$ given by $f(x) := \begin{cases} 3 & 0 \leq x < \frac{1}{3} \\ 1 & \frac{1}{3} \leq x < \frac{2}{3} \\ 3 & \frac{2}{3} \leq x \leq 1 \end{cases}$. The function f is non-monotonic on $[0,1]$.

The upper Riemann integral of f is

$$\overline{\int_{[0,1]}} f = \inf \left\{ \int_{[0,1]} g : g \text{ is a piecewise constant on } [0,1] \text{ and } g(x) \geq f(x) \ \forall \ x \in [0,1] \right\}$$

$$= \frac{7}{3}$$

Similarly, the lower integral of f is given by

$$\underline{\int_{[0,1]}} f = \sup \left\{ \int_{[0,1]} g : g \text{ is a piecewise constant on } [0,1] \text{ and } g(x) \leq f(x) \ \forall \ x \in [0,1] \right\}$$

$$= \frac{7}{3}$$

Since $\overline{\int_{[0,1]}} f = \underline{\int_{[0,1]}} f$, the function f is Riemann integrable on $[0,1]$ and $\int_{[0,1]} f = \frac{7}{3}$.

(d) $f \in \mathcal{R}[0,1]$ and f is neither monotone nor continuous.

Consider the function $f : [0,1] \to \mathbb{R}$ given by $f(x) := \begin{cases} 0 & x \in \{0,1\} \cup ([0,1] \setminus \mathbb{Q}) \\ \frac{1}{q} & x \in (0,1) \cap \mathbb{Q},\ x = \frac{p}{q},\ p,q \in \mathbb{N},\ \text{and} \\ & p,q \text{ are relatively prime} \end{cases}.$
We note that f is known as the Riemann function. Thus it is well known that this function is not piecewise continuous nor is it monotone.

6. Prove or justify, if true or provide a counterexample, if false.

(a) If $f(x) \leq g(x) \leq h(x)$ for all $x \in [a,b]$, and $f, h \in \mathcal{R}[a,b]$, then so is $g \in \mathcal{R}[a,b]$.

This is true by the *Squeeze Theorem.*

(b) If $f \in \mathcal{R}[a,b]$, then f is continuous on $[a,b]$.

This is a false statement. Consider $f : [0,3] \to \mathbb{R}$ given by $f(x) := \begin{cases} 2, & 0 \leq x \leq 1 \\ 3, & 1 < x \leq 3 \end{cases}$
Then we have that $\int_0^3 f(x) = 8$, and thus $f \in \mathcal{R}[0,3]$, but f is not continuous.

(c) If $|f| \in \mathcal{R}[a,b]$, then $f \in \mathcal{R}[a,b]$.

This is a false statement. Consider the function $f : [0,1] \to \mathbb{R}$ given by $f(x) := \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$, and let $P := \{x_0, x_1, \ldots, x_n\}$ be any partition of $[0,1]$. Then $M_i = 1$ and $m_i = -1$ for all $i = 1, 2, \ldots, n$. thus $U(f,P) = 1$ and $L(f,P) = -1$ for all P. Thus $U(f) = 1$ and $L(f) = -1$. Thus f is not integrable.

However, $|f|(x) = 1$ for all $x \in [0,1]$. Since $|f|$ is a continuous function $|f|$ is integrable on $[0,1]$.

(d) Let f be bounded on $[a,b]$. If P and Q are partitions of $[a,b]$, then $P \cup Q$ is a refinement of both P and Q.

This is a true statement because this satisfies the definition of a refinement, since both $P \subseteq P \cup Q$ and $Q \subseteq P \cup Q$.

(e) If f is continuous on $[a,b)$ and on $[b,c]$, then $f \in \mathcal{R}[a,c]$.

This is a false statement. Consider the function $f : [0,5] \to \mathbb{R}$ given by $f(x) := \begin{cases} \frac{x}{x-2}, & 0 \leq x < 2 \\ 0, & 2 \leq x \leq 5 \end{cases}$. Since an asymptote exists and is unbounded on $[0,5]$, we have that f is not Riemann integrable.

(f) If $f, g \in \mathcal{R}[a,b]$, then $f - g \in \mathcal{R}[a,b]$.

This is true by *Theorem 7.1.5 c*, since it can be rewritten as $f + (-g)$.

(g) If f is monotone on $[a,b]$, then $f \in \mathcal{R}[a,b]$.

This is a true statement by *Theorem 7.2.6*:

Theorem. If $f : [a,b] \to \mathbb{R}$ is monotone on $[a,b]$, then $f \in \mathcal{R}[a,b]$.